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MAKRO

MAKRO Model Documentation

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Beta version of the MAKRO model

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[1]Peter Bache designed the bargaining problem.

1 Introduction

1.1 Foreword

MAKRO is a large scale macroeconomic model of the Danish economy with short and long run predictive capabilities. There are four economic agents in the economy; households, firms, the government, and the foreign agents demanding Danish exports. As in most modern macroeconomic models, the behavior of households and firms are microfounded and forward looking.[2] Government behavior, however, follows a set of exogenous rules, and exports are determined by a demand function which incorporates aspects of different models of trade.

These different agents interact in the labor market, the capital market and the product market, and a key component of the work done in the model is the characterization of how these markets work. In particular, the relationship between long run and short run behavior in the economy results from the nature of frictions, such as the price setting behavior of firms and the staggered nature of the wage bargaining process, which affect these markets.

Following a shock or a policy intervention, the model converges after a period of cyclical normalization when short-run frictions fade. The convergence path, and particularly how the model reacts to temporary versus permanent shocks, is determined by the forward looking nature of optimal decisions.

Convergence happens towards a long run path which is theoretically defined and empirically determined from demographic, educational and socioeconomic conditions. This path is the model's forecast of the Danish economy when not affected by short run frictions, and it is fundamental for policy evaluation. Due to continuing movements in demographics and other exogenous factors the model is not in steady state, neither initially nor in the long run. Instead, it converges to a moving long run solution.

MAKRO differs from DSGE models in that it is a deterministic perfect foresight model. DSGE models solve for optimal decisions which are functions of state variables and contain the information pertaining to the stochastic nature of the model. These optimal decision functions are defined in a neighborhood of the model's steady state. MAKRO is instead a computational general equilibrium model which solves for a single path for all its variables. This solution relies on a set of initial and terminal conditions and reflects all policy changes and variations in exogenous factors one wishes to study.

MAKRO also differs from other models of its type due to its size. The household side of the model solves a model of overlapping generations each with a life cycle of 85 years, and the firm side of the model currently solves for 9 different sectors in the economy. It is a nonlinear model with a large number of endogenous variables, and as a professional planning and budgeting tool, the model's variables must correspond exactly to their counterparts in the data.

The model has a large set of parameters which are either estimated with econometric methods, calibrated from data, or taken from existing literature. Here we bring into the Computational General Equilibrium framework standard econometric methodology from DSGE models such as impulse response matching. Calibrating and estimating the large number of parameters requires large volumes of data which are obtained mainly from Denmark's register data. Of all data, population plays the most important role as it is the main exogenous input in the model.

One of the main purposes of MAKRO is to characterize the government budget balance, and how it responds to shocks and policy changes. This requires a considerable disaggregation of the fiscal part of the model as it must be able to evaluate a large num-

[2]Household and firm decisions are supported by a detailed specification of their objective functions, preferences, technology, and budget sets.

ber of tax and transfer interventions as well specific public consumption changes. This detail is mirrored in the life cycle detail of household consumption, savings and employment needed to accurately generate income tax revenues. It is also present in the sectoral disaggregation of production and the choice of inputs within each sector, as well as in the interactions between sectors described in the input-output structure of the economy, all of which are necessary to determine value added and corporate taxes.

All this detail has an important collateral benefit as it allows for the aggregation of heterogeneous micro responses to shocks or policy changes, resulting in a better characterization of the aggregate effects and fiscal implications of both.

The model represents work in progress. Many details may change during the next few years, but the overall structure is in place. The documentation is also work in progress and will evolve alongside the model.

1.2 This Documentation

The documentation contained in the subsequent chapters is a description of the model version as of November 2021. Although it is written mainly for model users to have an understanding of the background for the computational code, each chapter contains a description of the relevant theoretical part which can be understood by a wider audience.

Households. There are two types of households in the model. Optimizing households and Hand-To-Mouth (HTM) households.

Optimizing households solve a dynamic life cycle problem within an overlapping generations model. They maximize utility by choosing optimal non-durable consumption and savings into liquid assets, optimal housing, and optimal job search effort and hours worked. Within non-durable consumption they decide also on the optimal composition of a consumption bundle. The consumption/savings decision is dynamic and forward looking. Households choose the total amount of liquid non-housing net financial assets and this total volume of wealth is allocated to different assets in a portfolio composition estimated from the data. The optimal housing decision is also dynamic and forward looking, and reflects costs of mortgage financing, of housing depreciation and housing maintenance, as well as capital gains from house prices and revenues from land sales. The optimal choice of the non-durable consumption bundle is a static decision organized in a sequence of cost minimization problems. The optimal job search decision is also a dynamic forward looking decision.

Hand-to-mouth agents have zero financial assets and allocate their income between non durable consumption and housing every period. This is a proxy for an explicit model of financial constraints. The presence of HTM households helps aggregate consumption track income over the life cycle, and increases the aggregate marginal propensity to consume out of income shocks, as changes in income are fully transmitted to expenditure for these agents. The proportion of HTM agents in the model is estimated to match these targets in the data.

Household members die in our model, and when they die they leave bequests, which have associated warm glow utility. Bequests received are taken as given by the optimizing agent. The mapping between bequests given and received at different ages is an allocation matrix estimated from the data and which enters the model exogenously.

Production and Price Setting. Domestic output is produced by private firms and by the government.

There are eight private production sectors in the economy corresponding to eight broad classes of goods and services. In each sector firms use capital, labor, and materials (intermediate inputs) to produce output. Quantities of inputs are combined in a production function to produce units of ouput. Capital is subject to a time to build constraint of one period which makes investment decisions forward looking, and to investment adjustment costs which makes the optimal decision dynamic. Capital goods can be purchased from multiple supplying sectors, and from both domestic and foreign sources. Employment adjustment is also forward looking and subject to frictions. Firms incur costs from posting vacancies which are filled with a probability that is outside the firm's control. Optimal use of materials is a static decision, and, like capital goods, these can also be purchased from multiple supplying sectors, and from both domestic and foreign sources. Firms are price takers in input markets.

Private firms do not only make decisions regarding optimal use of inputs. They also set prices to maximize firm value. Price setting behavior occurs taken optimal input decisions as given and is an independent part of the model relative to the rest of firm optimization. The price setting problem adds price-adjustment costs to a monopolistic competition model of varieties. The resulting price adjustment is slow and forward looking.

Public production differs from its private counterpart and is detailed in conjunction with the chapters on government.

Labor Market. The model of the labor market contains heterogeneous households and firms. Households of different ages choose the supply of hours and optimal search effort. Labor demand comes from firms in different sectors posting vacancies optimally. A matching technology brings together vacancies and workers searching for jobs. The market closes with bargaining between unions representing workers and firms which sets the market wage. Wage rigidity is introduced via staggered wage bargaining.

Exports. Exports are modeled using a reduced form which incorporates insights from various models of trade, as well as mirroring the determinants of imports generated by MAKRO. The Export demand equation includes a measure of the size of the export market, a price ratio measure of our competitiveness in this market, a measure of domestic output, and lagged exports. The price elasticities of export demand in the different exported goods are fundamental parameters in MAKRO, as in any small open economy model. They are a key source of concavity in an otherwise largely linear model and allow the model to have a finite solution. For that reason a significant effort and care has been taken in the econometric estimation of these parameters. Details of the econometric work are available in additional documentation.

Government and public production. One key purpose of MAKRO is to determine the government budget balance, and how it responds to shocks and policy changes.

From an accounting perspective the government budget balance consists of government income minus government expenditure. Government expenditure consists mainly of government consumption and income transfers. These are tightly linked to demographics, to employment levels and, due in part to regulatory constraints, to wage levels. Government income consists mainly of taxes and duties. The main tax component is the personal income tax which depends on both employment and the wage level. Corporate taxes depend on firm earnings. Duties depend on the level and composition of aggregate demand.

From an economic perspective, the government produces goods and therefore we need a theory of public production. This differs from its private counterpart in that it is not built around a specification for the production function but rather on the value of the inputs into production. In the public sector these consist of depreciation costs, wage payments, and costs of intermediate inputs. The economic approach is important because the government is a large employer and this has an impact on the overall equilibrium in the economy, but also because some of the uses of inputs are part of a planned public agenda (for example in planned investment) which can be forecasted and in this way impact on the short term behavior of the model.

Input/Output Structure. The Input-/Output system is the collection of market clearing conditions, where the demand for materials, private consumption, government consumption, investment, and exports is met by supply from domestic and foreign producers. The supply side of the IO structure is given by 9 domestic and 9 foreign producing sectors. Some of these will have zero quantities if for example there are zero purchases from foreign construction sectors or from foreign public goods providers. The demand side ultimately also consists of the same 9 sector level of disaggregation. However, demanded goods have heterogeneous degrees of intermediate aggregation. Investment into capital goods by firms is sourced from only a handful of producing sectors, consumption goods demanded by households are intermediate aggregations of the 9 produced goods into 5 different consumption goods, and exported goods are also different reorganizations of the 9 goods produced at the bottom of the tree of the economy.

These mappings, for example between the definition of the 5 consumption goods demanded by households and the 9 different production sectors, can be viewed not just as demand coming directly from households and into the different production sectors through layers of nested sub-utility functions, but differently as layers of zero profit markets/firms that transform the basic goods into the upper level goods the agents demand. This transformation then occurs via a constant returns to scale "technology" which generates the necessary prices. Due to this equivalence, the lower demand-nest levels from households and firms are coded and contained in the IO computer files, and, as this is a very dense part of the model, their description is present in both places (in the household, firm, export, etc, chapters, as well as in the IO chapter).

Finally, at the very bottom of the demand side construction is the decomposition between domestic production and imports which is given by a constant elasticity of substitution aggregator. At this level there is substitutability between domestic and foreign supply in response to price changes. The prices at this level are the most disaggregated prices in the model, and it is at this level that taxes are included.

Calibration and Estimation. Every chapter contains some description of how we find values for the respective parameters. The document "The empirical basis for MAKRO" contains additional descriptions of the different procedures. Most parameters are calibrated using available data (over 1,500 in the latest version), so that the model is consistent with the national accounts. Most of these are "level parameters" such as the scale parameters in CES functions, which ensure that MAKRO hits the right levels for the data-covered endogenous variables. The vast majority of calibrated parameters is determined using a single relation/equation, and this relationship is static. Solving for these parameters using data is our static calibration procedure. It yields time series of these parameters for the available historical data period. Other parameters are determined using dynamic relationships such as forward looking first order conditions. These parameters are recovered in our dynamic calibration procedure. Before performing dynamic calibration we need to forecast some parameters obtained in static calibration.

The static calibration process generates historical time series for the different parameters. These time series can display structural trends such as a growing service sector. They also capture short run fluctuations and structural breaks. This information is treated econometrically with ARIMA models in order to generate forecasts of parameter values. With these in hand we can then solve the forward looking equations to recover the associated parameters.

Finally, some parameters are closely related to short run fluctuation behavior. These parameters are estimated by shocking the model and comparing the resulting impulse responses in artificial data with those obtained from SVAR models estimated on actual data. This is a standard methodology in DSGE models which we bring into our CGE framework.

1.3 Computational MAKRO

MAKRO is coded in GAMS which is an efficient software for solving large scale systems of nonlinear equations.

1.3.1 Notation

One problem that arose was that of having a system to name the large number of variables and parameters in the model. Notation in the documentation is consistent with the code but not identical. In the code nearly all objects have long descriptive names which allow for their identification in a dense computational environment. The code names are mostly in Danish because the users of this code will be Danish, while in the documentation the working language is English as the model is meant to be understood by a universal audience.

Some simple organizational choices are made for names in the code: quantities have prefix q, prices have prefix p and nominal values have prefix v. Many variables are recognizable in the code using common sense: K is associated with capital, L with labor, C with consumption, Y with output, etc.

In the documentation most object names are much shorter to ease notation while Greek letters are used for parameters following the academic literature common use. As an example a depreciation rate will be labeled δ in the documentation while having a long name in the code. One Greek letter pervasive in the documentation is μ. This character denotes usually share parameters which are a part of the widely used CES tree approach in production and consumption and in the code it is replaced by the letter u. While in the documentation μ will be used identically in different chapters without risk of confusion, in the code u will have additional characters and indices added to provide identification.

One other aspect of variable name organization is the naming of the same object at different levels of aggregation. This can be done by extending the variable name for example to aggregate or consider an age specific quantity, or by using the same name with additional indexing. For example a superset $s*$ can contain not just the nine items pertaining to the nine different production sectors in s, but also different subsets of the elements in the set s, allowing for various degrees of aggregation without changing the name of a variable.

One important aspect of the code, and one important capability of GAMS is the ability to organize the data using indices and sets. As the model has a large number of demand side and supply side items, identification of such items occurs through appropriate set description and indexing. For example, an object such as $q[d, s, t]$ will denote the quantity q demanded by sector d and supplied by sector s at time t.

The most important sets are time (t), which currently runs from 2000 to 2099, age (a), which currently runs from 16 to 100, and the non-numerical set, s, which currently has nine values identifying eight private sectors and one public sector. Additional sets are used to index capital goods, consumption goods, export goods, and intermediate inputs. Of these, the last three sets (consumption (c), exports (x), intermediate inputs (r)) are demand side reorganizations of the nine sector production set s. The index for capital goods covers machinery, buildings and inventories and is an independent set.

1.3.2 Code organization

The code is divided into different modules which reflect the theoretical chapters mentioned here. The modules can be solved separately, but each requires inputs from and provides outputs to other modules.

The code modules are: Consumers and Household Income, Finance and Private production, Pricing, Labor market, Exports, Public production, Government, Government expenses and Government revenues, Input-Output, Taxes, and the module Aggregates.

2 Households

Households choose optimal amounts of savings and expenditure, and within the expenditure choose the different types of goods they consume. A particularly important good is housing. The model must replicate several important features of the data. First moments include aggregate levels and life cycle profiles of housing ownership, mortgage debt, non housing wealth, and non housing consumption. The peak of home ownership occurs around age 60 in the data and the average household holds little non housing wealth until the mid 40's, after which wealth accumulation explodes. Of the many higher order moments, the most fundamental one is the marginal propensity to consume out of an income shock. All these issues require specific features of the model.

2.1 Basic Definitions

The model is a discrete time, perfect foresight, overlapping generations model of the life cycle. The full size of the cohort aged a in period t is given by $N_{a,t}$ and this quantity is exogenous and obtained from the data. There are two types of households, the financially constrained and the unconstrained, and this is a permanent state in that a constrained household is constrained in its entire life cycle, with the same being true for unconstrained households. A fraction Υ of households are constrained in their savings and borrowing activity. They are the "hand to mouth" consumers. As in Campbell and Mankiw (1989) these agents spend their entire income every period. The remaining, unconstrained, fraction $1-\Upsilon$ is able to access bond and asset markets at no cost.

The timing convention is that all decisions are taken, income is realized, and consumption occurs at the end of each period. The household problem for each type is to choose an optimal consumption path over the life cycle given its income path. The income path is endogenous as the household decides also on its participation in the labor market, but that choice is discussed in the labor market chapter. Furthermore, consumption of different non-housing goods is the result of a CES nest optimization sequence which relates to the input-output structure of the data, and this is also detailed elsewhere.

In terms of exposition this chapter is closely linked to the labor market chapter. In the text the following symbols are generally used with the associated purposes: η will denote an elasticity, δ a destruction or depreciation rate, τ will denote a tax rate, and θ will be the preference discount rate, with β being a discount factor. Υ is the fraction of hand to mouth consumers.

2.1.1 Age, Utility, and Survival Rates

Individuals live up to age A, and the age index runs $a = 0, 1, 2..., A$, where the index value $a = 1$ refers to the first age of life when children are born and until they become one year old. The first index value $a = 0$ is reserved for an initial condition for children in the model. Consumption and income flows of an individual aged a during period t are written $c_{a,t}$ and $y_{a,t}$. The stock of accumulated non-housing net financial assets B are defined at the end of the period as the result of the current period's decisions and are written $B_{a,t}$, so that $B_{a-1,t-1}$ are assets determined at the end of the previous period and carried over to the current period t when the agent is one year older. The variable B excludes mortgages and pension wealth but includes any non-mortgage bank debt incurred in the process of buying a house.

Households derive flow utility $U_{a,t}$ from non durable consumption $c_{a,t}$ and from housing services arising from the end of period stock of owned housing $D_{a,t}$.[3] This utility flow

[3]Our model of housing has its roots in the model of durables by Mankiw (1982). We do not define explicitly the utility function of hand-to-mouth agents as we detail below.

is a CES function and has a habit component as we detail in the appendix. Define the partial derivatives

$$U^c_{a,t} = \frac{\partial U_{a,t}}{\partial C_{a,t}}, \quad U^d_{a,t} = \frac{\partial U_{a,t}}{\partial D_{a,t}}$$

Unconstrained households also derive utility from owning wealth itself, $V^{Wealth}_{a,t}$, and separately they also have warm glow utility $V^{Beq}_{a,t}$ from leaving their assets as bequests in case of death. These assets will consist of any financial assets $B_{a,t}$ plus any equity on housing available at the time of death. Constrained households die and leave housing in bequests, but since, as we detail below, they make no optimal decisions, there is no need to define wealth or bequest utility functions for them.

We define the survival rate $s_{a,t}$ to be the probability an individual aged a at time t will be alive and making decisions at time t+1, one year older.

2.1.2 Migration

The population of a given age at a point in time will generally be such that

$$N_{a,t} = N_{a-1,t-1}s_{a-1,t-1} + I_{a,t} - E_{a,t}$$

where some agents will have either left, $E_{a,t}$, or entered, $I_{a,t}$, the country at this point.

We make the necessary assumptions to ensure that those entering the country have the same consumption, income, housing and employment as surviving residents, otherwise the model would have an intractable amount of heterogeneity. On the other hand, those leaving take with them their assets. As for housing, agents leaving sell their housing stock while agents entering come in with zero housing, such that the total amount of housing in the country is unchanged by immigration, and retains its characteristic of being a good that is not traded across borders.

2.1.3 Budget constraint

The budget constraint of any individual household of type j, aged a, can be written as

$$B^j_{a,t} = B^j_{a-1,t-1} + r^j_{a,t}B^j_{a-1,t-1} + y^j_{a,t} - p^c_t c^j_{a,t} - f\left(D^j_{a,t}, D^j_{a-1,t-1}\right)$$

$$B^j_{a^{ini}-1,t-1} = \bar{B}^j$$

The object $B^j_{a^{ini}-1,t-1}$ denotes non housing net financial assets carried over from childhood and available at the first optimizing age a^{ini} which is 18 years of age. This is a quantity $\bar{B}^j$ calculated from the data and detailed in the subsection on children below.

Received bequests are included in income $y^j_{a,t}$. Households receive bequests from, and leave bequests to, both constrained and unconstrained agents. Income includes wages, taxes, transfers, and pension payments.

The object f captures all elements of the budget constraint that relate to owned housing. The non durable consumption price p^c_t is a CES aggregate price which is the same for all types and ages as the CES consumption tree is assumed to be the same for all types and ages. Prices contain taxes and/or subsidies implicitly. The index j will be omitted from this point onwards unless required for clarity.

2.2 Optimization

A financially constrained household has no net financial assets, $B_{a,t} = 0$. Its budget constraint is given by

$$0 = y_{a,t} - p^c_t c_{a,t} - f\left(D_{a,t}, D_{a-1,t-1}\right)$$

These households do have an optimal allocation decision between housing and non durable consumption. We do not model it explicitly and instead approximate it with the following relationship

$$D_{a,t} - \chi^D D_{a-1,t-1} = \lambda^D_{a,t} \cdot \left(C_{a,t} - \chi^C C_{a-1,t-1}\right) \cdot \left(\frac{P^D_t}{P^C_t}\right)^{-\eta}$$

where in the years with available data $\lambda^D_{a,t}$ is an exogenous calibration object, the χ are habit parameters, and η is an elasticity of substitution. This then changes in the forecast period and when we shock the model where $\lambda^D_{a,t}$ is held constant.

2.2.1 Unconstrained households: savings decision

Unconstrained households choose both non durable consumption and housing sequences optimally to maximize the discounted present value of utility flows. The optimal decision for B is given now. The dynamic first order conditions can be obtained mechanically by replacing the consumption variable with the budget constraint in the sequence problem, and choosing end of period assets at every age. We obtain

$$U^c_{a,t}\frac{1}{p^c_t} = \frac{1}{p^c_{t+1}}\frac{R^B_{a+1,t+1}}{1+\theta}U^c_{a+1,t+1}s_{a,t} + \frac{1}{1+\theta}s_{a,t}\frac{\partial V^{Wealth}_{a,t}}{\partial B_{a,t}} + \frac{1}{1+\theta}\left(1-s_{a,t}\right)\frac{\partial V^{Beq}_{a,t}}{\partial B_{a,t}}$$

where $R^B_{a+1,t+1}$ is a marginal rate of return,

$$R^B_{a+1,t+1} = \frac{\partial}{\partial B_{a,t}}\left\{\left(1+r_{a+1,t+1}\right)B_{a,t}\right\}$$

The household trades-off current with future marginal utility of consumption. On the left hand side, the last unit of income used for current consumption yields $1/p^c_t$ units of consumption with marginal utility $U^c_{a,t}$. Optimality implies this must be identical to what is obtained from alternatively saving this marginal unit of income, earning a gross marginal return R, and using it next period for consumption, taking into account that one may die. This is given by

$$\left\{\frac{1}{p^c_{t+1}}U^c_{a+1,t+1}\right\}R_{a+1,t+1}$$

weighed by the survival rate $s_{a,t}$ and discounted by the factor $\frac{1}{1+\theta}$ to match the current marginal utility. In addition, if you survive you will also derive the utility of the ownership of the extra wealth. On the other hand, in the small chance $(1-s_{a,t})$ that you die, you get the marginal change in bequest utility, $\partial V^{Beq}_{a,t}/\partial B_{a,t}$, which is measured in the future and discounted back for mechanical consistency, as in case of death the agent only dies tomorrow (and therefore after the current savings decision).

Last period of life

The household lives up to 100 years of age, as we need to truncate the model. The survival rate is therefore zero in the final age, $s_{A,t} = 0$, but bequests still occur. With this parameter at zero we obtain

$$U^c_{A,t}\frac{1}{p^c_t} = \frac{1}{1+\theta}\frac{\partial V^{Beq}_{A,t}}{\partial B_{A,t}}$$

and this condition determines assets at the end of life. However, setting the survival rate at zero induces an abrupt change in behavior at the end of life that distorts the optimal choice due to the truncation of life. We instead use the following equation where the survival rate is the actual rate observed at 100 years of age, $s_{A,t} \neq 0$, and where we replace the would-be consumption of 101 year olds with the consumption of this period's 100 year olds:

$$U^c_{A,t}\frac{1}{p^c_t} = \frac{1}{1+\theta}\left\{\frac{1}{p^c_{t+1}}s_{A,t}R_{A,t}U^c_{A,t} + s_{A,t}\frac{\partial V^{Wealth}_{A,t}}{\partial B_{A,t}} + (1-s_{A,t})\frac{\partial V^{Beq}_{A,t}}{\partial B_{A,t}}\right\}$$

2.2.2 Unconstrained households: housing choice

Housing is a durable stock variable and an element in the optimal choice of overall consumption. Like savings, the choice of housing is a dynamic forward looking decision with an associated intertemporal first order condition. The general expression for this condition is[4]

$$U^c_{a,t}\frac{1}{p^c_t}\left(\frac{\partial f_t}{\partial D_{a,t}}\right) = U^d_{a,t} + \frac{s_{a,t}}{1+\theta}U^c_{a+1,t+1}\frac{1}{p^c_{t+1}}\left(R^D_{a+1,t+1} - \frac{\partial f_{t+1}}{\partial D_{a,t}}\right)$$
$$+\frac{s_{a,t}}{1+\theta}\frac{\partial V^{Wealth}_{a,t}}{\partial D_{a,t}} + \frac{(1-s_{a,t})}{1+\theta}\frac{\partial V^{Beq}_{a,t}}{\partial D_{a,t}}$$

with

$$R^D_{a+1,t+1} = \frac{\partial}{\partial D_{a,t}}\left\{(1+r_{a+1,t+1})\,B_{a,t}\right\}$$

which reads: when you sacrifice $1/p^c_t$ units of non durable consumption today and use the money to buy extra housing, you have an immediate marginal utility loss from reduced consumption. This is the left hand side of the equation. On the right hand side you gain immediately the direct marginal utility of the durable good, $U^d_{a,t}$, and also tomorrow a gain of bequest utility if you die, and, if you don't, the marginal utility of non durable consumption associated with the effect of the additional housing bought today on tomorrow's income. This effect contains the income released due to the fact that less housing investment is needed tomorrow $\frac{\partial f_{t+1}}{\partial D_{a,t}}$. It contains also the impact of housing decisions on portfolio choices via R^D. This last effect helps characterize the user cost of housing in more detail as the household faces mortgage interest costs on the mortgage part, but opportunity costs on the non mortgage part. These opportunity costs now reflect also the change in portfolio weight on bank debt when the volume of housing changes. Finally, extra housing will generate extra wealth utility in the case that housing is included in the measure of wealth.

2.2.3 Putting the two together

It is useful to aggregate the two first order conditions. Here we merge them by eliminating the marginal utility of consumption in period t+1, $U^c_{a+1,t+1}$. This is useful because it yields the housing $D_{a,t}$ user cost expression measured at time t, and comparable to the current consumption price. We detail the resulting expression in the appendix, but for now we obtain

$$U^c_{a,t}\frac{1}{p^c_t}\left[\frac{\partial f_t}{\partial D_{a,t}} + \frac{1}{R^B_{a+1,t+1}}\frac{\partial f_{t+1}}{\partial D_{a,t}} - \frac{R^D_{a+1,t+1}}{R^B_{a+1,t+1}}\right] = U^d_{a,t}$$
$$+\frac{s_{a,t}}{1+\theta}\left[\frac{\partial V^{Wealth}_{a,t}}{\partial D_{a,t}} + \frac{\frac{\partial f_{t+1}}{\partial D_{a,t}} - R^D_{a+1,t+1}}{R^B_{a+1,t+1}}\frac{\partial V^{Wealth}_{a,t}}{\partial B_{a,t}}\right]$$

[4] Just as in the savings optimal choice, this equation needs to be adjusted in the final age of life.

$$+\frac{(1-s_{a,t})}{1+\theta}\left[\frac{\partial V_{a,t}^{Beq}}{\partial D_{a,t}}+\frac{\frac{\partial f_{t+1}}{\partial D_{a,t}}-R^D_{a+1,t+1}}{R^B_{a+1,t+1}}\frac{\partial V_{a,t}^{Beq}}{\partial B_{a,t}}\right]$$

where

$$USER_{a,t}=\underbrace{\left[\frac{\partial f_t}{\partial D_{a,t}}+\frac{1}{R^B_{a+1,t+1}}\frac{\partial f_{t+1}}{\partial D_{a,t}}-\frac{R^D_{a+1,t+1}}{R^B_{a+1,t+1}}\right]}_{\text{User Cost of } D_{a,t} \text{ measured at time t in nominal units.}}$$

which is a key object in the household problem. This merged equation has an intuitive reading. When the household sacrifices one unit of current consumption to buy additional housing, it must weight the direct marginal of housing $U^d_{a,t}$ against the loss of marginal utility of consumption $U^c_{a,t}$ net of the gain in the marginal utility of wealth and bequests.

We can also merge the two first order conditions by eliminating the current marginal utility of consumption $U^c_{a,t}$. This is useful because, given the assumptions we make regarding the form and content of the utility of wealth and bequests, it generates the following computationally friendly expression which we use in the code.

$$U^c_{a+1,t+1}\cdot\frac{s_{a,t}}{1+\theta}\cdot USER_{a,t}\cdot\frac{R^B_{a+1,t+1}}{p^c_{t+1}}=U^d_{a,t}$$

The intuition here is also clear. When we buy an additional unit of housing today we gain the corresponding direct marginal utility of housing. This must be identical to the marginal utility of consumption we could obtain tomorrow if instead of spending the money on housing we used that amount of money, $USER_{a,t}$, capitalized it $\frac{R^B_{a+1,t+1}}{p^c_{t+1}}$, and used it to eat then. We must also account for the survival rate and discount it back to today.

2.3 Children

Our consumer starts life as a teenager. The data reveals both income and assets for children below the optimizing age in the model, which is 18 years.[5] Fitting the budget constraint of these children is important as it allows us to correctly calibrate initial wealth at 18 years of age, and also to correct for otherwise excessive consumption of the associated parental household.

Rather than modelling children as optimizing agents, we let their consumption be implicit in the parent's problem and create an exogenous income transfer variable from parents to children that will fit the child's budget constraint at zero consumption and is just enough to hit the asset target at age 18. Children are born with zero assets and for a few ages they actually have recorded disposable income, so that their budget constraints are given by

$$B_{a,t}=B_{a-1,t-1}+r_{a,t}B_{a-1,t-1}+yDisp_{a,t}+Transfer_{a,t}$$

$$B_{0,t}=0$$

$$0<a<18$$

where the initial condition $B_{0,t}$ has an index zero for age, since it denotes any assets carried over from before the first age of life. Since any transfers are current flows and bequests received are included in the income variable this quantity must be zero.

[5] Optimal labor search decisions start at age 16, and this is possible because, by eliminating wealth effects from the labor decision we make the two problems independent of each other.

Total transfers received by children of a given age, $Transfer_{a,t} \times N_{a,t}$, are paid for by the adult cohorts that have children of that age, which we know from the data. We then take an equal amount from all parents so that one parent with for example a 13 year old child will pay the same amount to that child that every parent pays for a 13 year old child. Parents of different ages will have different numbers of children of various ages, and therefore across the age of parents the total amount spent in child transfers will vary.

As we calibrate this equation to the data, we obtain the value of initial assets for agents at the first optimizing age. Note that for the purpose of this document, the transfer from the parent to the child is hidden inside the disposable income variable of the parent. Finally, as this is a correction of income it applies to adult rule of thumb consumers as well as to adult optimizing agents.

Only children who grow up to be unconstrained agents are the beneficiaries of such transfers. Constrained children simply do not exist as we set their budget constraint to be identically zero at all ages until they start optimizing life at age 18 with zero assets.

2.4 Detailing the f housing object

The housing the household buys and sells is an object which aggregates "bricks" and land. The "bricks" part of the house is produced mostly with inputs purchased from the construction sector. The country's entire stock of land is held by households inside their housing good, and land available for the construction of new houses is the land released as a result of housing depreciation.[6] An intermediary then buys output from the construction sector as well as other intermediate inputs, and buys land from households released from depreciated housing, packages these together, and sells the resulting housing good back to households. Land is introduced in MAKRO in order to have a production factor in rigid supply. In reality Land is not a totally rigid factor and we allow for exogenous increases in the aggregate stock, but land prices are a key component of house price movements.[7]

Housing is also the overwhelming source of household debt, and housing finance is a major fraction of total financial activity. Houses here are financed with a mortgage with an age specific fraction of mortgage financing to house value, $\mu_{a,t}$.[8] This is a loan to value (LTV) constraint.[9] The object $\mu_{a,t}$ is exogenous to the household but it is modeled to include the effect of house prices. Therefore the model generates quantities for mortgage debt which change via the extensive margin (as the stock of housing changes) as well as via movements in prices when the extensive margin is constant. The modeling of $\mu_{a,t}$ is discussed in the appendix.

We introduce an exogenous supply of rental accommodation, H, with also an exogenous rent, to capture the non negligible amount of existing public and regulated rental housing. We do not model the link between the rental market and the owned housing market and therefore rent expenses appear only as an exogenous term in the budget constraint of the household, and rental housing does not yield utility.

Owned housing enters the budget constraint via the object f. This object is a cost function which contains costs with financing, taxation, and maintenance, and deducts revenues from downsizing and from land sales. We now detail the elements of f with extended algebra and proofs in the appendix. Define first net investment in housing of an agent aged a at time t as

$$z_{a,t} = D_{a,t} - (1 - \delta_t)\, D_{a-1,t-1}$$

[6] Only the bricks part of the house dies with depreciation. The corresponding land is sold. The depreciation rate δ is detailed in the appendix.

[7] Davis and Heathcote (2006), The Price and Quantity of Residential Land in the United States.

[8] Endogenous mortgage ratios make the current model too big and complex to solve.

[9] Kaplan, Mitman and Violante (2017) include loan to income (LTI) constraints.

where in the first optimizing age we have $z_{a,t} = D_{a,t}$. Then postulate the exogenous relationship for the mortgage debt stock $X^M_{a,t}$,

$$X^M_{a,t} = \mu_{a,t} P^D_t D_{a,t}$$

where $\mu_{a,t}$ is exogenous to the household. Combine now assets B, income, and rental housing in the auxiliary variable Δ:

$$\Delta_{a,t} \equiv B_{a,t} - (1 + r_{a,t}) B_{a-1,t-1} - \left(\underbrace{\tilde{y}_{a,t} - rent_t H_{a,t}}_{y_{a,t}} \right)$$

so that we obtain the budget constraint

$$\Delta_{a,t} + P^C_t C_{a,t} + f(D_{a,t}, D_{a-1,t-1}) = 0$$

In the appendix we show that mortgage payments and down-payments can be manipulated away from the budget constraint so that we get

$$\begin{aligned} f(D_{a,t}, D_{a-1,t-1}) = \left(1 + r^{mort}_t\right) \mu_{a-1,t-1} P^D_{t-1} D_{a-1,t-1} - \mu_{a,t} P^D_t D_{a,t} \\ + P^D_t D_{a,t} - P^D_t (1 - \delta_t) D_{a-1,t-1} \\ + \left(\tau^W_t + x^\delta_t\right) P^D_{t-1} D_{a-1,t-1} \\ - P^D_{t-1} D_{a-1,t-1} \alpha^{Land}_t \end{aligned}$$

Since f is a cost function we have non mortgage financing $(1 - \mu_{a,t})$, wealth taxes τ^W_t and maintenance costs x^δ_t, and mortgage interest payments r^{mort}_t, all with a positive sign. Carried over undepreciated housing, and income earned from selling land, are revenues and therefore appear with a negative sign. The appendix details the computation of the factor α^{Land}_t which defines the revenue from land sales. Collect now terms to get:

$$f(D_{a,t}, D_{a-1,t-1}) = (1 - \mu_{a,t}) P^D_t D_{a,t}$$

$$+ \left\{ \left(1 + r^{mort}_t\right) \mu_{a-1,t-1} + \tau^W_t + x^\delta_t - \frac{P^D_t}{P^D_{t-1}} \left(1 - \delta^d_t\right) - \alpha^{Land}_t \right\} P^D_{t-1} D_{a-1,t-1}$$

The partial derivatives of this expression which enter the user cost expression and the optimality condition are now trivial to compute and are all either exogenous to or taken as given by the household.

2.4.1 User cost and expectations

Expectations of future house prices have a significant effect on household decisions. As we have a perfect foresight model, we cannot introduce an internally consistent expectations mechanism, so we use a pragmatic approximation that allows us to modulate the effect of future house prices on current decisions. We postulate that agents expect the price to move in the correct direction but by only a fraction λ of the total magnitude:

$$E_t\left(P^D_{t+1}\right) = (1 + g_p) P^D_t + \lambda \left(P^D_{t+1} - (1 + g_p) P^D_t\right)$$

$$E_t\left(P^D_{t+1}\right) - P^D_t = (1 - \lambda) g_p + \lambda \left(P^D_{t+1} - P^D_t\right)$$

This then enters the first order condition through the term $\frac{\partial f_{t+1}}{\partial D_{a,t}}$ which is present in the user cost and in multiplying the wealth and bequests objects.

$$E_t\left(\frac{\partial f_{t+1}}{\partial D_{a,t}}\right) = \left\{\left(1+r_{t+1}^{mort}\right)\mu_{a,t} + \tau_{t+1}^{W} + x_{t+1}^{\delta} - \frac{E_t\left(P_{t+1}^{D}\right)}{P_t^D}\left(1-\delta_{t+1}^{d}\right) - \alpha_{t+1}^{Land}\right\} P_t^D$$

We extend this approach also to the land price inside the term α_{t+1}^{Land}. The expected user cost is then written

$$USER_{a,t} = \underbrace{\left[\frac{\partial f_t}{\partial D_{a,t}} + \frac{1}{R_{a+1,t+1}^{B}}\frac{\partial f_{t+1}}{\partial D_{a,t}} - \frac{R_{a+1,t+1}^{D}}{R_{a+1,t+1}^{B}}\right]}_{\text{User Cost of } D_{a,t} \text{ measured at time t.}}$$

or using the explicit derivatives - where $R_{a+1,t+1}^{B} \equiv 1 + r_{a+1,t+1}^{B}$ and $R_{a+1,t+1}^{D} \equiv r_{a+1,t+1}^{D}$:

$$USER_{a,t} = \frac{P_t^D}{1+r_{a+1,t+1}^{B}}\left[\left(1-\mu_{a,t}\right)r_{a+1,t+1}^{B} + \mu_{a,t}r_{t+1}^{mort} + \tau_{t+1}^{W} + x_{t+1}^{\delta} + \delta_{t+1}^{d}\right] +$$

$$-\frac{P_t^D}{1+r_{a+1,t+1}^{B}}\left[\left(\frac{E_t\left(P_{t+1}^{D}\right) - P_t^D}{P_t^D}\right)\left(1-\delta_{t+1}^{d}\right) + \alpha_{t+1}^{Land}\right] - \frac{r_{a+1,t+1}^{D}}{1+r_{a+1,t+1}^{B}}$$

where the interest rate terms contain the opportunity cost attached to the non mortgaged part of the house, the mortgage rate cost attached to the mortgage part, and the marginal effect on the portfolio return from changing the amount of housing. This last effect will increase the user cost of housing as long as the bank lending rate is high enough to make $r^D < 0$.

Hidden user costs. We add one term to the user cost measure, potentially unmeasured inside the maintenance cost x_t or the depreciation rate, or inside the cost of bank borrowing related to the house, r^D, for which we do not have an accurate measure. The idea is that houses can carry asymmetric information risk and also that we are missing actual transaction costs. In our implementation these raise the average user cost even though we do not account for them in the budget constraint, and significantly help the model fit the data.

Aggregate objects. Aggregates are constructed as: income $\sum_a N_{a,t}y_{a,t}$, consumption $\sum_a N_{a,t}^{j}C_{a,t}^{j}$, assets $\sum_a N_{a,t}^{j}B_{a,t}^{j}$, and housing $\sum_a N_{a,t}^{j}D_{a,t}^{j}$.

2.5 Bequests

Warm glow utility from bequests is fundamental for the model to be able to replicate the large amounts of wealth held at the late ages of the life cycle. Not only that, the shape of the bequest utility function also limits the level of debt households can incur during the young ages of the life cycle and this substitutes both for precautionary savings and for credit constraints in the model. The function describing direct utility of wealth shares these properties with the bequest utility function and we describe below what separates these two functions.

2.5.1 Death

The key property of death is that, in any given period, it occurs before the relevant decisions are taken. On January first of period t the agent is alive or dead. If he is alive he has to wait 365 days until December 31st to consume and save. On the other hand, if he is dead he has no more income and no longer consumes or saves, and his assets are distributed amongst his heirs as an exogenous income transfer. This transfer is only received on December 31st of period t.

2.5.2 Bequests received, liquidating housing, and bequests in utility

All agents leave bequests to and receive bequests from both constrained and unconstrained agents. Constrained agents leave zero net financial assets, but just like unconstrained ones leave considerable housing. In the event of death houses are sold and mortgages are liquidated, so that bequests received will consist of liquid assets plus the liquid value of the equity on the house after liquidation. Given the exogenous mortgage ratio relationship, in the event of death the equity that is transformed into liquid assets next period is given by

$$(1-\mu_{a,t})\, P^D_{t+1} D_{a,t}$$

This is then taxed and the resulting net value received by the multiple heirs.[10] From an accounting perspective, bequests given and received must add up to the same amount, corrected for taxes. The mapping from bequests given to bequests received is done with an allocation matrix M_t constructed from the data and detailed in the appendix.

Bequests do not just enter the budget constraint. They are a key object in preferences. The bequest utility of the dying agent is given by

$$V^{Beq}_{a,t} \equiv \xi^{0Beq}_a \frac{\left[X^{Beq}_{a,t}\right]^{1-\eta}}{1-\eta}$$

and now we define the interior object X as

$$X^{Beq}_{a,t} \equiv \left(1-\tau^{beq}_{t+1}\right) \frac{B_{a,t} + p^D_t\,(1-\mu_{a,t})\, D_{a,t} + V^{PensionB}_{a,t}}{p^C_{t+1}} + \xi^{1Beq}_a$$

It is important to note that, as this is a utility construction, there is a degree of freedom in the definition of the object $X^{Beq}_{a,t}$. Here we attach value to the sum of assets, rather than, for example, attaching a separate special value to the house, although both formulations are feasible.[11] Using the sum of assets, and including some pension entitlements inside X allows for substitutability so that the household is indifferent regarding which type of asset makes it richer (we use the same approach in the utility of wealth), or makes those who will inherit these bequests richer. The fundamental property to preserve is that it is a concave and increasing function. There is one small detail that deserves mention. We value the house inside X at the current price p^D_t. This is a pragmatic assumption which will be useful below. The derivatives of this utility function are given by

$$\frac{\partial V^{Beq}_{a,t}}{\partial B_{a,t}} = \left(1-\tau^{beq}_{t+1}\right) \frac{1}{p^C_{t+1}} \Omega^{Beq}_{a,t}$$

[10] Higher liquidation costs of houses relative to liquid assets in case of death can be a significant incentive to substitute away from housing at the end of life, and help explain the downsizing pattern observed from around age 60 onwards. However, as we cannot currently observe these costs, we leave them out.

[11] Li, Liu, Yang, and Yao (2016) have a CES function of housing and other assets as bequest utility. Kaplan, Mitman and Violante (2017) do as here.

$$\frac{\partial V_{a,t}^{Beq}}{\partial D_{a,t}} = \left(1 - \tau_{t+1}^{beq}\right) \frac{(1 - \mu_{a,t})\, p_t^D}{p_{t+1}^C} \Omega_{a,t}^{Beq}$$

with

$$\Omega_{a,t}^{Beq} \equiv \xi_{a,t}^{0Beq} \left[X_{a,t}^{Beq}\right]^{-\eta}$$

The utility associated with bequests is parameterized with $\xi_{a,t}^0$ and $\xi_{a,t}^1$. The interior parameter $\xi_{a,t}^1$ will be strictly positive in some ages to accommodate the possibility of negative total assets at death, which is a possibility at the first young ages. An upper bound on $\xi_{a,t}^1$ implies a lower bound on combined assets. This of course implies an upper bound on debt, acting as precautionary savings in the model.

2.6 Merging the first order conditions

As we mentioned above, it is useful to detail the merging of the first order conditions as it will help us obtain a simpler and more insightful expression. Let

$$V_{a,t}^{Wealth} \equiv \xi_a^{0Wealth} \frac{\left[X_{a,t}^{Wealth}\right]^{1-\eta}}{1-\eta}$$

$$X_{a,t}^{Wealth} \equiv \frac{B_{a,t} + p_t^D\,(1 - \mu_{a,t})\, D_{a,t} + V_{a,t}^{PensionW}}{p_{t+1}^C} + \xi_a^{1Wealth}$$

with

$$\mathit{\Delta}_{a,t}^{Wealth} \equiv \xi_{a,t}^{0Wealth} \left[X_{a,t}^{Wealth}\right]^{-\eta}$$

and note that the elasticity η is the same in both the wealth and bequest utility objects. Again here only total assets including some pension entitlements matter. Now work the wealth and bequest utility parts of the merged expression to obtain

$$\frac{s_{a,t}}{1+\theta} \frac{1}{p_{t+1}^C} \left[(1 - \mu_{a,t})\, p_t^D + \frac{\frac{\partial f_{t+1}}{\partial D_{a,t}} - R_{a+1,t+1}^D}{R_{a+1,t+1}^B}\right] \mathit{\Delta}_{a,t}^{Wealth}$$

$$+\frac{(1 - s_{a,t})}{1+\theta} \frac{\left(1 - \tau_{t+1}^{beq}\right)}{p_{t+1}^C} \left[(1 - \mu_{a,t})\, p_t^D + \frac{\frac{\partial f_{t+1}}{\partial D_{a,t}} - R_{a+1,t+1}^D}{R_{a+1,t+1}^B}\right] \Omega_{a,t}^{Beq}$$

where we have the user cost inside both these expressions which yields

$$\equiv +\frac{1}{1+\theta} \frac{USER_{a,t}}{p_{t+1}^C} \left\{s_{a,t} \mathit{\Delta}_{a,t}^{Wealth} + (1 - s_{a,t}) \left(1 - \tau_{t+1}^{beq}\right) \Omega_{a,t}^{Beq}\right\}$$

so that merging the two first order conditions results in

$$U_{a,t}^c \left[\frac{USER_{a,t}}{p_t^c}\right] - \left[\frac{1}{1+\theta} \frac{USER_{a,t}}{p_{t+1}^C}\right] \left\{s_{a,t} \mathit{\Delta}_{a,t}^{Wealth} + (1 - s_{a,t}) \left(1 - \tau_{t+1}^{beq}\right) \Omega_{a,t}^{Beq}\right\} = U_{a,t}^d$$

and this has an intuitive meaning. The gain in marginal utility of housing equals the marginal utility loss from lower consumption net of the marginal gain from higher wealth, where the marginal utilities are weighted by the price ratio so that all is measured in the same units at time t.

Final details. The pension entitlements inside bequest utility and wealth utility differ. Regarding bequest utility, some pension plans cease at death, while others contain an insurance element and continue to pay descendants and $V_{a,t}^{PensionB}$ is the amount from pensions paid to descendants in case of death. Regarding utility from wealth, once households reach a certain age they can freely choose the payout schedule from 'Kapitalpension' and 'aldersopsparing'. Many choose not to withdraw anything until the latest possible date, 20 years after retirement, when the whole amount is paid out at once. These pension types are therefore close to stocks and bonds in terms of liquidity, and they are treated the same way in the utility function so we do not get an unreasonable jump in utility when the whole amount is converted to other assets 20 years after retirement. Therefore the object $V_{a,t}^{PensionW}$ differs from the bequest utility pension object. In both cases we also account for the fact that hand-to-mouth agents also have some types of pensions and these have to be decoupled from the total pension wealth by age in the economy.

One last detail concerns the fact that we do not account for transaction costs incurred in liquidating or trading houses (or other assets). It is possible to account for these through a tax-like term in the bequest case, while some other form can be used in wealth. However, our objects of interest are utility objects and they are included in the model to help match the high volume of assets at old age and the almost absence of negative net wealth at young ages, as well as ensuring we obtain a sensible discount rate for the household. This last detail is discussed at the end of the document.

2.7 Household Income

The budget constraint of the household is given by

$$B_{a,t} = B_{a-1,t-1} + r_{a,t}B_{a-1,t-1} + y_{a,t} - f\left(D_{a,t}, D_{a-1,t-1}\right) - P_t^C C_{a,t}$$

The income term $y_{a,t}$ incorporates a large number of taxes and transfers as well as the exogenous expenditure in rental housing. Before we detail the different elements inside $y_{a,t}$ it is useful to briefly define the rest of the items in the budget constraint.

Wealth B denotes non housing net financial assets and excludes pension wealth. It includes ownership of financial stocks and bonds, as well as bank deposits, and subtracts non-mortgage bank debt. The object f contains all items of the budget constraint that relate to owned housing and consists of total net expenditure on owned housing. The term $P_t^C C_{a,t}$ denotes all non-housing consumption expenditure. Consumption prices include taxes. The rate of return on wealth $r_{a,t}$ is a portfolio rate of return.

2.7.1 Income

The income variable $y_{a,t}$ contains the following elements: labor market income from employment and non employment, $y_{a,t}^W$, net pension income, $y_{a,t}^{PY} - y_{a,t}^{PC}$, expenditure on rental housing, and net taxes and transfers.[12]

$$y_{a,t} = y_{a,t}^W + y_{a,t}^{PY} - y_{a,t}^{PC} - R_t^{rent} H_{a,t} + T_{a,t}^{Net}$$

Net taxes and transfers $T_{a,t}^{Net}$ contain an assortment of income transfers $T_{a,t}^Y$, various taxes not related to housing or pensions $T_{a,t}^{\tau}$, received bequests $T_{a,t}^{Beq}$, net income flows associated with children $T_{a,t}^{children}$, and residual items.[13]

[12]Labor market income is $\left(1-\tau_t^w\right) h_{a,t}\rho_{a,t}w_{a,t}\left[q_{a,t}^e + \mu_{a,t}^u\left(1-q_{a,t}^e\right)\right]$ where $h_{a,t}\rho_{a,t}w_{a,t}$ is the wage per hour per productivity unit, and $\left(q_{a,t}^e, \mu_{a,t}^u\right)$ are respectively the fraction of time employed and the replacement ratio for the non-employment benefit. More detail can be found in the labor market chapter.

[13]The object $T_{a,t}^{children}$ in the code is: vBoernFraHh[a,t] - vHhTilBoern[a,t].

$$T^{Net}_{a,t} = T^{Y}_{a,t} - T^{\tau}_{a,t} + T^{Beq}_{a,t} + T^{children}_{a,t} + T^{other}_{a,t}$$

Of all these different items, only labor market income is endogenous to the household as it results from a decision of how much to engage in the labor market.

The tax object $T^{\tau}_{a,t}$ captures a large number of specific taxes.[14] Income taxes, local taxes, property taxes, taxes on financial income from stocks, taxes on income from individually held companies, estate taxes, labor market specific taxes, etc.[15] These taxes are grouped differently depending on the purpose. For example, wealth taxes on property are removed and included in the housing term f.

2.7.2 Different income definitions

In the budget constraint we can define income in a variety of ways. We defined above the income variable $y_{a,t}$ excluding gross financial income and excluding terms related to owned housing. This is convenient from the point of view of handling the model and its first order conditions for optimality. However, there are other ways of defining income which relate better to the data.

Financial income

First we can add financial income to the initial income variable:

$$B_{a,t} = B_{a-1,t-1} + \underbrace{r_{a,t}B_{a-1,t-1} + y_{a,t}}_{\text{including net financial income}} - f\left(D_{a,t}, D_{a-1,t-1}\right) - P^{C}_{t}C_{a,t}$$

The return on assets $r_{a,t}B_{a-1,t-1}$ that we use in the budget constraint is a gross return, $r_{a,t}B_{a-1,t-1}$, so that the taxes paid on financial income $\tau r_{a,t}B_{a-1,t-1}$, are included in the term $T^{\tau}_{a,t}$ inside $y_{a,t}$. The sum $r_{a,t}B_{a-1,t-1} + y_{a,t}$, is then an object that contains net (of taxes) financial income.[16]

The reason we do not directly work with a tax rate on interest earnings in the budget constraint is that income taxation incorporates all income (interest income, wages, etc) and applies a tax rate on the total. One cannot, without further assumptions, identify the tax on returns. Therefore, it is gathered in the budget condition. However, we need a marginal tax rate on returns for our first order conditions. And the return is a portfolio return with stocks, bonds, deposits and bank debt. There it is assumed that stocks are taxed at an average of the high and low share return tax rate (where the high is weighted at 0.2), while for tax on bonds and bank deposits it is assumed that they have the same tax rate as the average for the current cohort (the marginal tax rate varies by age because the fraction of population paying top tax varies with age).

Other items

Similarly, inside the housing object f we have wealth taxes $\tau^{W}_{t}P^{D}_{t-1}D_{a-1,t-1}$ which effectively reduce disposable income. We could then write

$$B_{a,t} = B_{a-1,t-1} + \underbrace{r_{a,t}B_{a-1,t-1} + y_{a,t} - \tau^{W}_{t}P^{D}_{t-1}D_{a-1,t-1}}_{\text{including net financial income and removing wealth taxes}} - \hat{f}_{a,t} - P^{C}_{t}C_{a,t}$$

$$\hat{f}_{a,t} = f\left(D_{a,t}, D_{a-1,t-1}\right) - \tau^{W}_{t}P^{D}_{t-1}D_{a-1,t-1}$$

Taxes are not the only items in the housing object which are expenses carried over from the previous period and which reduce disposable income before any decisions can be taken

[14] In the code this object is constructed with the variables vtHh[aTot,t] and vtHhx[a,t].

[15] In the code: vtDirekte[t], vtKilde[t], vtBund[a,t], vtTop[a,t], vtKommune[a,t], vtEjd[a,t], vtAktie[a,t], vtVirksomhed[a,t], vtDoedsbo[a,t], vtHhAM[a,t], vtPersRest[a,t].

[16] Note that the marginal return on assets, which enters the dynamic optimality condition, is the after tax return.

in period t. Other expenses are housing maintenance, x_t, and mortgage interest payments r_t^{mort} on the fraction of the house mortgaged $\mu_{a-1,t-1}$, which reduce disposable income as they are firm prior commitments. On the other hand the income associated with land sales on depreciated property increases disposable income.

In our model of the household the full disposable income before decisions are taken is therefore

$$\underbrace{r_{a,t}B_{a-1,t-1} + y_{a,t} - \left[\tau_t^W + r_t^{mort}\mu_{a-1,t-1} + x_t - \delta_t \frac{\alpha_t^{Land}}{P_{t-1}^D}\right] P_{t-1}^D D_{a-1,t-1}}_{\text{including net financial income and removing wealth taxes, mortgage interest, and maintenance, and adding land sales}}$$

Finally, from the point of view of the data, rental housing expenses are a consumption decision, and so while in the model they are a lump sum item, in the data they are not a part of disposable income. We can then write

$$\underbrace{r_{a,t}B_{a-1,t-1} + y_{a,t} + R_t^{rent}H_{a,t} - \left[\tau_t^W + r_t^{mort}\mu_{a-1,t-1} + x_t - \delta_t \frac{\alpha_t^{Land}}{P_{t-1}^D}\right] P_{t-1}^D D_{a-1,t-1}}_{\text{including NFI, excluding rental housing, removing wealth taxes, mortgage interest, and maintenance, adding land sales}}$$

2.7.3 Income of HTM households

These have no assets so their budget constraint is

$$0 = y_{a,t} - f\left(D_{a,t}, D_{a-1,t-1}\right) - P_t^C C_{a,t}$$

Income of HTM households is not model consistent. Taxes on capital income and wealth taxes are included in $T_{a,t}^{\tau}$ but cannot be removed without further assumptions imposed on the data. The same is true for taxes on interest income and interest expenses as they are part of taxes on personal / taxable income and cannot be identified. On the other hand taxes on income from stocks can be removed. So, the income of HTM agents is identical to that of optimizing agents with the following correction

$$y_{a,t}^{HTM} = y_{a,t} + T_{a,t}^{Stocks}$$

2.8 Pensions

Pension income enters the disposable income of households as an exogenous income quantity, and pension wealth satisfies accumulation consistency requirements which are also exogenous to the household.

MAKRO uses a simplified version of the detailed pension model in DREAM. The data is taken from the DREAM pension model and aggregated into three pension types: 1) pensions that have already been taxed (alderspension, index label 'Alder'), 2) capital pension (kapitalpension, index label 'Kap') taxed with a flat rate, and 3) the aggregate of other pensions, taxed when received by households (ratepensioner, livrentepensioner and ATP, with index label 'PensX').

As MAKRO does not distinguish between gender within a cohort, we sum pension contributions paid by men and women into their pension funds, as well as pensions received by men and women.

2.8.1 Pension wealth

The three diferent types of pensions, indexed by j, are modelled as three separate actuarially fair pension schemes. The law of motion for individual pension wealth $B_{a,t}^{P,j}$ in a given pension scheme j is similar to the one for net financial assets in the household:

$$B_{a,t}^{P,j} = \left(B_{a-1,t-1}^{P,j} + \underbrace{r_{a,t}^{P,j} B_{a-1,t-1}^{P,j}}_{TR_{a,t}^{P,j}\text{: Total Return}} \right) \frac{N_{a-1,t-1}}{N_{a,t}} + y_{a,t}^{PC,j} - y_{a,t}^{PY,j}$$

The stock of pension wealth $B_{a,t}^{P}$ is the amount of wealth in the pension fund available to distribute as pension income to a recipient of a given cohort. The object $y_{a,t}^{PC}$ denotes pension contributions which are payments made by households into the pension fund. The object $y_{a,t}^{PY}$ denotes pension income which are payments made by the pension fund and received by households.[17]

Pension wealth is corrected for population changes to ensure the entire pension wealth is distributed and the pension fund does not go bankrupt. The aggregate pension wealth of the household is given by the sum over the pension types j:

$$B_{a,t}^{P} = \sum_j B_{a,t}^{P,j}$$

and the aggregate pension wealth of a given pension fund j is given by

$$B_t^{P,j} = \sum_a N_{a,t} B_{a,t}^{P,j}$$

and the index j will be ignored unless it is deemed useful in an explanation. The object $B_t^{P,j}$ is an asset for households and a liability for the pension fund. The pension fund is a zero profit vehicle so that its assets equal its liabilities to households.

2.8.2 Pension Contributions and Pension Income

It is assumed that an exogenous part of wages is paid as pension contributions to each type of pension. The object $y_{a,t}^{PC}$ is such that

$$y_{a,t}^{PC} = \lambda_{a,t}^{PC} \cdot w_{a,t}$$

The parameter $\lambda_{a,t}^{PC}$ is calibrated so the pension contribution matches the pension data from DREAM.

It is also assumed that an exogenous age specific share of the primo pension wealth is paid out and received by households each period as pension income $y_{a,t}^{PY}$ such that

$$y_{a,t}^{PY} = \lambda_{a,t}^{PY} \times B_{a-1,t-1}^{P}$$

The parameter $\lambda_{a,t}^{PY}$ is calibrated so the pension income received by households matches the pension data from DREAM.[18]

The entire pension system is calibrated such that all pension contributions are eventually paid out to the household, and this takes into account the fact that we truncate the life span to 100 years of age.

Figure 1 at the end of this document shows the cross section of contributions and income in 2016 for PensX.

[17] In the code the different objects are labelled as follows: $B_{a,t}^{P,j} = vHh[pens, a, t]$, with contributions $y_{a,t}^{PC,j} = vPensIndb[pens, a, t]$ and pension payments $y_{a,t}^{PY,j} = vPensUdb[pens, a, t]$. The pension type index j is $pens = (PensX, kap, Alder)$. Total return is $TR_{a,t}^{P,j} = vHhPensAfk[pens, a, t]$.

[18] In the code we have for each pension type j: $\lambda_{a,t}^{PY,j} \equiv rPensUdb[j, a, t]$, $\lambda_{a,t}^{PC,j} \equiv rPensIndb[j, a, t]$.

2.8.3 Finite lives

Death before age 100. Unlike the household budget constraint where the assets of the dead are given away as bequests, here the pension assets of the dead are managed by the pension fund, and are redistributed as a bonus to pension recipients. Therefore, the object $y_{a,t}^{PY}$ contains this bonus payment. To make this point clearer we can write the law of motion again and separate "normal" income $\tilde{y}_{a,t}^{PY}$ from the "death bonus":

$$y_{a,t}^{PY} N_{a,t} = \tilde{y}_{a,t}^{PY} N_{a,t} + \underbrace{\left(1 + r_{a,t}^{p}\right) B_{a-1,t-1}^{P} \left(1 - s_{a-1,t-1}\right) N_{a-1,t-1}}_{\text{death bonus}}$$

Going back to the law of motion of pension assets

$$B_{a,t}^{P} = \left(1 + r_{a,t}^{p}\right) B_{a-1,t-1}^{P} \frac{N_{a-1,t-1}}{N_{a,t}} + y_{a,t}^{PC} - \underbrace{\tilde{y}_{a,t}^{PY} + \left(1 + r_{a,t}^{p}\right) B_{a-1,t-1}^{P} \left(1 - s_{a-1,t-1}\right) \frac{N_{a-1,t-1}}{N_{a,t}}}_{\text{Total Pension Income per Person}}$$

which we can write

$$B_{a,t}^{P} = \left(1 + r_{a,t}^{p}\right) B_{a-1,t-1}^{P} \left[s_{a-1,t-1} \frac{N_{a-1,t-1}}{N_{a,t}} \right] + y_{a,t}^{PC} - \underbrace{\tilde{y}_{a,t}^{PY}}_{\text{"Normal" Income per Person}}$$

Truncation at age 100. Pension accumulation is modeled to replicate observed pension wealth stocks and flows at all ages, including those older than 100 years. Thus, pension wealth does not vanish at age 100. As in the model there are no surviving 101-year olds the actuarial fairness in the model is closed by paying the terminal wealth as a balloon payment to the 100 year olds. Therefore, at the terminal age

$$0 = \left(B_{A-1,t-1}^{P,j} + TR_{A,t}^{P,j} \right) \frac{N_{A-1,t-1}}{N_{A,t}} + y_{A,t}^{PC,j} - \underbrace{\left(y_{A,t}^{PY,j} + B_{A,t}^{P,j} \right)}_{\text{Total Payment}}$$

The pension fund does not disappear even though cohorts die. Aggregate pension wealth (end of period) in the pension fund is then given by

$$B_t^P = \sum_{a}^{A-1} N_{a,t} B_{a,t}^P$$

As a final remark, not all pension types run until age 100. Some pension schemes end at an age prior to age 100 and therefore the algebra above applies to the pension-specific terminal age.

2.8.4 Composition and returns of pension portfolio

The aggregate pension wealth of the pension fund B_t^P is invested in stocks and bonds. The pension fund portfolio structure is a simpler version of the household portfolio. Assets of a specific type i held by the pension fund j, $A_{i,t}^{P,j}$, are an exogenous fraction of total wealth:[19]

$$A_{i,t}^{P,j} = \omega_{i,t}^{P,j} \cdot B_t^{P,j}$$

The financial portfolio of the pension sector is assumed to be independent of the different type of pensions (capital, taxed, non-taxed) it consists of, and so $\omega_{i,t}^{P,j} \equiv \omega_{i,t}^{P}$ has no

[19] Assets of a specific type are indexed by i = bonds, domestic equity and foreign equity.

pension type index j, and so $A^{P,j}_{i,t} = \omega^P_{i,t} \cdot B^{P,j}_t$.[20] The return on pension wealth is then independent of the type of pension (except for the adjustment terms in the historical period to match data).

The return consists of two terms: an interest rate r^P_t and a revaluation rate r^{RP}_t.[21] The interest rate for the pension sector consists of weighted average for interests for bonds and dividends for equity in its asset portfolio:

$$r^{P,j}_t = \frac{\sum_i r_{i,t} A^{P,j}_{i,t}}{\sum_i A^P_{i,t}} + J^{P,r,j}_t = \underbrace{\frac{\sum_i r_{i,t} \cdot \omega^p_{i,t}}{\sum_i \omega^p_{i,t}}}_{\text{average portfolio interest rate } r^P_t} + J^{P,r,j}_t$$

where if an asset is a stock (i=stocks) , the rate is the (observed) dividend rate

$$r_{stocks,t} = \frac{DIV_{stocks,t}}{V_{stocks,t-1}}$$

The revaluation rate on pension sectors assets are also given by a weighted average with an adjustment term:

$$r^{RP,j}_t = \frac{\sum_i r^{RP}_{i,t} \cdot A^{P,j}_{i,t}}{\sum_i A^{P,j}_{i,t}} + J^{P,rev,j}_t = \underbrace{\frac{\sum_i r^{RP}_{i,t} \cdot \omega^p_{i,t}}{\sum_i \omega^p_{i,t}}}_{\text{average portfolio revaluation rate } r^{RP}_t} + J^{P,rev,j}_t$$

such that differences across pension types are captured in the J term. The revaluation rate in the case of stocks is the capital gains rate.[22]

The individual adjustment terms for interest and capital gains are a measure of the deviation between the average rate and the observed rate. As we use the average rates, we also capture these individual adjustment terms in a joint term for total returns. The total return on pension wealth is given by:

$$TR^{P,j}_{a,t} = \left(1 - \tau^P_t\right)\left(r^P_t + r^{RP}_t\right) \cdot B^{P,j}_{a-1,t-1} + J^{TRP}_{a,j,t}$$

where τ^P_t is the effective tax rate on pension returns and $J^{TRP}_{a,j,t}$ is a pension type-and-age-specific adjustment term that ensures that the age-specific return matches the data (DREAM's pension data). The interest r^P_t and revaluation r^{RP}_t terms are the average terms construted above and are the same for all pension types j. The differences in total return across pension types are then absorbed by the adjustment term $J^{TRP}_{a,j,t}$ which is common to the interest and capital gains objects. The final condition is that the sum of adjustment terms for all cohorts equals zero on the total of all pension types j.[23]

$$\sum_{\{a,j\}} J^{TRP}_{a,j,t} = 0$$

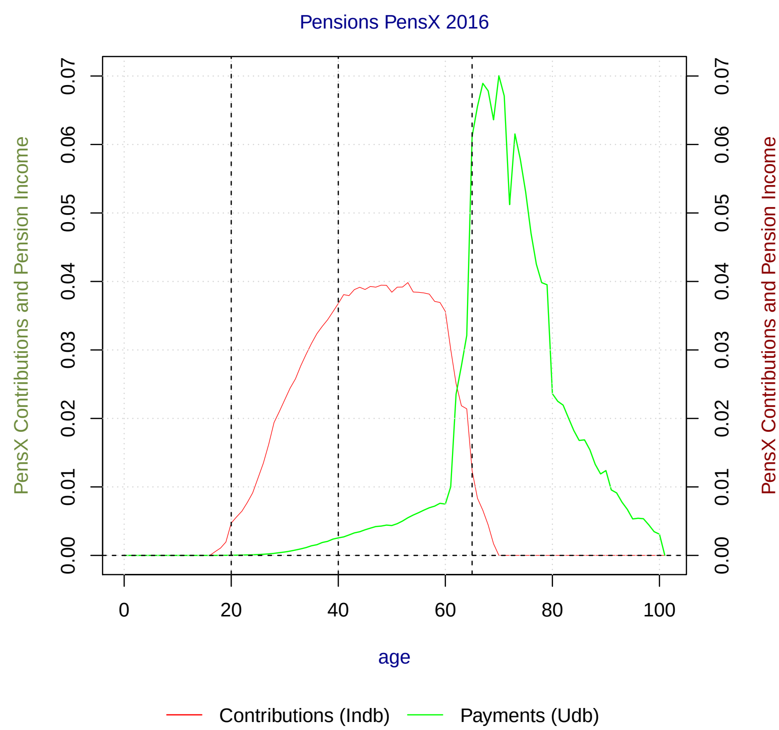
The reason pension returns are the same for all different pension types is that we assume all pension firms have the same portfolio.

[20] Where $i \in \{Bonds, Equity, ForeignEquity\}$

[21] A revaluation is a capital gain when assets are not traded so the gain remains an acounting gain.

[22] The capital gains rate takes into account new stock issues. New stocks issues are exogenous to the model. The value of the firm determined endogenously in MAKRO is only the fundamental part of the firm. That is the value added generated by hiring production factors and actually producing and selling output.

[23] In the code the different objects are labelled as follows: $\tau^P_t \equiv ft^{PAL}_t \cdot t^{PAL}_t$, $r^P_t \equiv r^{InterestPension}_t$, $J^{RP}_{a,t} \equiv J^{Return}_{pension,a,t}$, and $r^{RP}_t \equiv r^{RevaluationsPension}_t$.



Pensions PensX 2016
PensX Contributions and Pension Income
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0
20
40
60
80
100
age
Contributions (Indb)
Payments (Udb)

2.9 Household's Financial Portfolio

The budget constraint of a household is

$$B_{a,t} = (1 + r_{a,t}) B_{a-1,t-1} + y_{a,t} - P_t^C c_{a,t} - f(D_{a,t}, D_{a-1,t-1})$$

and only financially unconstrained households have non zero net non housing financial assets, $B \neq 0$.

The model only generates endogenously the financial variable B, but in the data this quantity is made up of the sum of different assets (stocks, bonds and bank deposits) minus the sum of liabilities (bank debt), so that $B = A - L$. This decomposition of B into assets and liabilities displays systematic patterns over the life cycle, and here we detail how to capture these features and use them in our model. We first look at a very simple example to highlight the approach.

2.9.1 Assets and Liabilities as functions of B

The exogenous portfolio composition is estimated from the data as in the following example with one asset and one liability. Assets A are related to net financial wealth B through the equation

$$A_{a,t} = I_A + \lambda B_{a,t}$$

and we have the same for liabilities

$$L_{a,t} = I_L + \phi B_{a,t}$$

and as $A_{a,t} - L_{a,t} = B_{a,t}$ we must have that our estimated parameters obey $I_L = I_A$ and $\lambda - \phi = 1$. As it turns out these properties are ensured by the mechanics of OLS estimation. The OLS regressions are $A = X\beta_A + \epsilon_A$, and $L = X\beta_L + \epsilon_L$ where A and L are column vectors. Taking the estimator for assets, $\hat{\beta}_A = (X'X)^{-1}X'A$, and using the fact that $A = B + L$, we can write $\hat{\beta}_A = (X'X)^{-1}X'B + \hat{\beta}_L$. Here, since the matrix X contains the vector B the OLS algebra implies

$$\hat{\beta}_A - \hat{\beta}_L = (X'X)^{-1}X'B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which are exactly the restrictions we need. These properties extend to the case where the regressions have explantory variables so that the matrix X has more columns. We therefore estimate these equations using OLS.[24] Then for historical data we add the orthogonal OLS error to the estimated regression so as to replicate the portfolio data exactly. For forward looking simulation we leave the orthogonal error (which has mean zero) out and use the estimated parameters, plus endogenous B and any endogenous explanatory variables inside X to generate a portfolio going forward.

2.9.2 General Structure

We only run regressions along the age dimension so that from here on the time index is ommitted. For several asset types i, and liability types j, we have at any given moment:

$$N_a A_a^i = I_a^i + \lambda^i N_a B_a$$

$$N_a L_a^j = I_a^j + \phi^j N_a B_a$$

[24]OLS is adequate as the relationships we are estimating are not a behavioural model. They are instead a way to capture the patterns observed in the data more acurately than just using averages as done in DREAM.

where I_a^i and I_a^j are the intercept functions for each age and (λ^i, ϕ^j) are the parameters associated with net assets. We have defined the variables in the regression to be the cohort totals $(N_a B_a, etc)$ instead of individual household quantities B_a. The intercept terms I_a are general functions of age and of auxiliary variables with aggregate net assets B as a scaling factor. With a first order polinomial in age and with one auxiliary variable Z the intercept would look as follows

$$I_a^{i,j} = I_0^{i,j}\left[\frac{B}{n}\right] + I_1^{i,j}\left[\frac{B}{S^a}\right]a + I_z^{i,j}\left[\frac{B}{Z}\right]N_a Z_a$$

with $n = \sum_a 1$, and $S^a = \sum_a a$, and $B = \sum_a N_a B_a$ and $Z = \sum_a N_a Z_a$. We can also have several Z variables to fit the portfolio.

The OLS regression objects Y and X are then (for asset i) given by $Y = \left[N_a A_a^i\right]$ with X containing four column variables with as many rows as ages a:

$$X = \left[\begin{array}{cccc} \frac{B}{n}, & \frac{B}{S^a}a, & \frac{B}{Z}N_a Z_a, & N_a B_a \end{array}\right]$$

2.9.3 Homogeneity

The way the intercept terms are defined plays a role in ensuring homogeneity of degree 1 in the model. Homogeneity of degree 1 is ensured if when increasing all exogenous variables by a common factor Λ the model yields all endogenous variables factored by the same Λ such that no relative quantities change. Consider the equation for asset A^i and aggregate over all ages to obtain

$$A^i = \sum_a I_a^i + \lambda^i B$$

and inserting the intercept we have

$$A^i = \sum_a \left\{ I_0^i\left[\frac{B}{n}\right] + I_1^i\left[\frac{B}{S^a}\right]a + I_z^i\left[\frac{B}{Z}\right]N_a Z_a \right\} + \lambda^i B$$

which, after cancelling terms, yields

$$\frac{A^i}{B} = \left[I_0^i + I_1^i + I_z^i + \lambda^i\right] = \text{constant}$$

The portfolio structure only needs to ensure the homogeneity of individual assets and liabilities with respect to B. The rest of the MAKRO model must then ensure B is homogeneous with respect to all other variables.

2.9.4 Remarks

The estimation procedure and the exact specification of these equations are agnostic (given the specification assumed) with respect to the data. There may be theoretical reasons to think portfolio composition should vary over the life cycle. If the data contains such heterogeneity, the estimated parameters will reflect that by having different values for different assets. In addition, the estimated portfolio is an optimal portfolio, because the underlying assumption is that agents made optimal decisions that resulted in what we observe. As the entire household problem generates endogenous variation for B and Z, the estimated portfolio model allows for endogenous variation of its constituent parts which by design is an optimal portfolio adjustment. Note also that any sluggishness in portfolio adjustment relative to economic conditions is already included in the estimated equations, either through lower coefficients attached to explanatory variables or through a higher intercept. Finally, the presence of the constant term yields an estimation error which is orthogonal to the life cycle, a property which is very useful in the forecasting role of the model.

2.9.5 Marginal returns

Given a portfolio structure we now must fit the budget constraint on historical data. The budget constraint with explicit assets and liabilities is

$$B_{a,t} = B_{a-1,t-1} + \underbrace{\left[\sum_i r_t^i A^i\left(B_{a-1,t-1}\right) - \sum_j r_t^j L^j\left(B_{a-1,t-1}\right)\right]}_{\text{Realized Total Return}} + ...$$

and, given *observed/realized* rates of return, it is completely characterized. The realized return on assets is

$$\underbrace{\left(\sum_i r_t^i I_{a-1,t-1}^i - \sum_j r_t^j I_{a-1,t-1}^j\right)}_{\text{intercept and attached rates}} + \underbrace{\left(\sum_i r_t^i \lambda_{t-1}^i - \sum_j r_t^j \phi_{t-1}^j\right)}_{\text{marginal return on B}} B_{a-1,t-1}$$

where we note again that the parameters and intercept functions are timed in the same way as the underlying assets they describe, but the rates of return are timed one period forward.

The **marginal** rate we are looking for in the Euler equation is then

$$R_{a,t}^B = R_t^B = 1 + \bar{r}_t^B = 1 + \left(\sum_i r_t^i \lambda_{t-1}^i - \sum_j r_t^j \phi_{t-1}^j\right)$$

and $\bar{r}_t^B$ is not age dependent since the parameters ϕ and λ are not age dependent and we assume that rates of return r_t^i or r_t^j on any assets and liabilities of unconstrained agents are not age related. Note that interest rates on bank debt may well be age related but we rule that out.

This is not the only marginal rate. If the auxiliary variable is endogenous there will be a marginal rate given by

$$\bar{r}_t^Z = \frac{B_{t-1}}{Z_{t-1}}\left(\sum_i r_t^i I_{z,t-1}^i - \sum_i r_t^j I_{z,t-1}^j\right)$$

This is the case for housing. Since bank debt is related to housing purchases, we select the housing stock $D_{a,t}$ (or housing value $V_{a,t}^D = P_t^D D_{a,t}$) as an auxiliary variable $Z_{a,t}$. As the portfolio is related to the housing stock, the choice of housing now influences the savings decision through its impact on portfolio composition and returns. Note that as the household changes its decision on housing D and on net financial assets B, the portfolio adjusts within the model as the data suggests it should. This adjustment is still exogenous as optimal portfolio composition is implicit in the estimated parameters of the portfolio structure. The additional marginal rate is then

$$\bar{r}_t^D = \frac{B_{t-1}}{D_{t-1}}\left(\sum_i r_t^i I_{d,t-1}^i - \sum_i r_t^j I_{d,t-1}^j\right)$$

and is generally non zero, unless the rate of return on assets and liabilities is the same. Note that if we want to write this derivative using house values instead of quantities we need to write it as

$$\bar{r}_t^D = \frac{B_{t-1}}{V_{t-1}^D} P_{t-1}^D \left(\sum_i r_t^i I_{d,t-1}^i - \sum_i r_t^j I_{d,t-1}^j\right) \equiv \frac{B_{t-1}}{D_{t-1}}\left(\sum_i r_t^i I_{d,t-1}^i - \sum_i r_t^j I_{d,t-1}^j\right)$$

This marginal rate helps characterize the user cost of housing in more detail as the household faces mortgage interest costs on the mortgage part, but opportunity costs on the non mortgage part. These opportunity costs now reflect also the change in portfolio weight on bank debt when the volume of housing changes.

2.9.6 The user cost of housing

We derived the expression

$$USER_{a,t} = \underbrace{\left[\frac{\partial f_t}{\partial D_{a,t}} + \frac{1}{R^B_{a+1,t+1}}\frac{\partial f_{t+1}}{\partial D_{a,t}} - \frac{R^D_{a+1,t+1}}{R^B_{a+1,t+1}}\right]}_{\text{User Cost of } D_{a,t} \text{ measured at time t.}}$$

and with

$$R^B_{a+1,t+1} \equiv R^B_{t+1} = 1 + \left(\sum_i r^i_{t+1}\lambda^i_t - \sum_j r^j_{t+1}\phi^j_t\right)$$

where

$$R^D_{a+1,t+1} \equiv \bar{r}^D_{t+1} = \frac{B_t}{V^D_t}P^D_t\left(\sum_i r^i_{t+1}I^i_{d,t} - \sum_i r^j_{t+1}I^j_{d,t}\right)$$

and we have[25]

$$USER_{a,t} = \underbrace{\left[\frac{\partial f_t}{\partial D_{a,t}} + \frac{1}{R^B_{a+1,t+1}}\frac{\partial f_{t+1}}{\partial D_{a,t}} - \frac{B_t}{V^D_t}P^D_t\frac{\left(\sum_i r^i_t I^i_{d,t} - \sum_i r^j_t I^j_{d,t}\right)}{1+\left(\sum_i r^i_t\lambda^i - \sum_j r^j_t\phi^j\right)}\right]}_{\text{User Cost of } D_{a,t} \text{ measured at time t.}}$$

2.9.7 Shocks and data

Shocks. Each different asset or liability has its own reward, and, in the absence of shocks to the model, realized and "expected" returns are identical. Since MAKRO is a perfect foresight model, when a shock occurs it changes the environment from one probability 1 scenario to a different probability 1 scenario. In the impact period of the shock (and only then), domestic stock returns (and only those) will differ from "expected" returns. Realized returns are always included in the budget constraint. Expected returns (which obey arbitrage conditions in the absence of shocks) are always included in the intertemporal first order conditions.

Data. As households in MAKRO are divided in 100 age groups, it is a requirement of the data set used to calibrate households that it contains data distributed across those age groups.[26] The task that MAKRO will be used for also requires that the sum of the wealth profiles over age correspond to the totals found in the national accounts. Such a data set was not available prior to the creation of the MAKRO life cycle profiles.

The administrative data used to create the wealth profiles is drawn from the Statistics Denmark's administrative data on wealth, with some additional data being drawn from the Lovmodel database. Aggregate data on wealth is drawn from the national accounts. Returns are based on aggregate data and the portfolio composition implied by the created asset profiles.

[25] The parameters λ and ϕ are in the code $dvHh2dvHhx$ and, in the code the $I^i_{d,t}$ and $I^j_{d,t}$ are called $dvHh2dvBolig['asseti', t]$ (inside the aldersprofiler.gms file).

[26] Reference: Christian P. Hoeck (2020). "The creation of lifecycle profiles for households in MAKRO."

The assets profiles are created using two steps: First a correspondence between the administrative data and the asset structure in MAKRO is established. Most of the asset and liability types in MAKRO have clear correspondences to the administrative data. This includes bank debt and deposits, real estate, mortgages, and bonds. In MAKRO stocks are divided into foreign and domestics stocks, but Statistics Denmark's wealth data only contains information on the combined value of stocks. Data from the Lovmodel database is therefore used to divide the combined value of stocks into foreign and domestic stocks. In the second step the asset and liability profiles are then scaled proportionately to match the aggregate values from the national accounts.

Rates of return are calculated based on aggregate values from the national accounts. Combining the rates of return with the created asset and liability age profiles, results in age profiles for total returns.

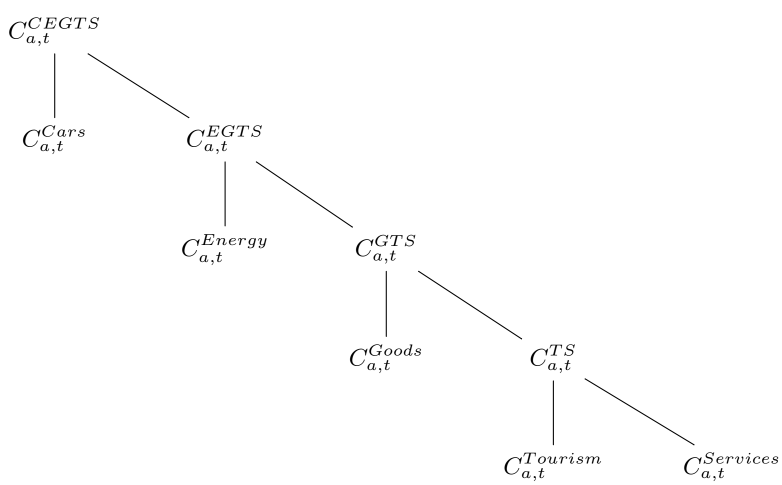
2.10 Consumption components

At the top of the household utility function we have two goods: owned housing and the non-housing consumption aggregate. The utility function is everywhere a CES function combining goods. Owned housing is a single good with no subcomponents. Non-housing consumption, on the other hand, aggregates many elements through a CES tree structure. Note, however, that rental housing is not an element in the CES tree, but instead it is an exogenous element in the budget constraint of the household.

The optimal choice of total consumption, savings, and housing, is described in the household chapter. In this chapter we detail the determination of the components of total non-housing consumption, $C_{a,t}$. The first decomposition of this object contains five different goods which are organized in the upper part of the tree. Household demand for these five consumption goods is a part of total demand for output from the nine domestic sectors as well as for imported goods, a process described in the Input/Output chapter.

2.10.1 Upper tree

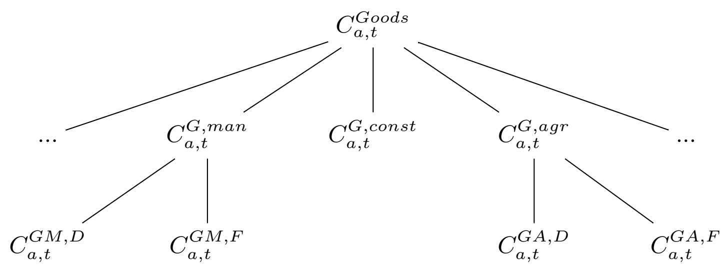
Within the utility function the different types of consumption come together in the following CES nest structure where non durable consumption of an agent aged a at time t is given by $C_{a,t} \equiv C_{a,t}^{CEGTS}$:



2.10.2 Lower tree

At the end of each branch we have specific consumptions. These consumptions are complex objects because they can be commanded from domestic or foreign sources, and because they can aggregate output from different sectors. The multisectoral composition of consumption components is a necessary result from the decomposition of the production side of the model into 8 private sectors plus the public sector, which must be allocated into the 5 consumption components above. Take the consumption of goods as an example. These goods can be produced in the manufacturing sector, in the agricultural sector, or in other sectors, and, to use two specific products as examples, not all beer is produced in Denmark, and not all apples are Danish.

The lower tree is organized in a specific sequence with the allocation of the nine production sectors into the five consumption goods in level 1 (on top) and the decomposition between domestic goods and imports in level 2 (at the bottom). This is a hypothetical example of the lower tree for $C_{a,t}^{Goods}$ where we see manufacturing, construction and agriculture in level 1:



All five consumption components have the same lower tree, although not all components have all branches. As an example there is no contribution of agriculture to the combined consumption object “cars”. This object consists mostly of manufacturing and services on the production side. Services here include sales, freight, and other services, and make up around 30% of the consumption object “cars”. Manufacturing makes up (most of) the remaining 70% and within that most of it is imported manufacturing as the cars themselves are not made in Denmark.

Having described the shape of the tree, we can now describe the optimization sequence that applies to the tree.

2.10.3 CES optimization

The approach of nested CES cost minimization is described in detail in the production chapter. The problem here is identical, only simpler as there are no extra elements such as technological progress or variable utilization multiplying consumption quantities. We can summarize the problem at every level of the consumption tree as follows

$$\text{Utility} \quad \Rightarrow \quad C^{ij} = \left[\left(\mu^i\right)^{\frac{1}{\eta}} \left(C^i\right)^{\frac{\eta-1}{\eta}} + \left(\mu^j\right)^{\frac{1}{\eta}} \left(C^j\right)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

$$\text{Derivative} \quad \Rightarrow \quad \frac{\partial C^{ij}}{\partial C^i} = \left(\mu^i \frac{C^{ij}}{C^i}\right)^{\frac{1}{\eta}}$$

$$\text{Demand/F.O.C.} \quad \Rightarrow \quad C^i = \mu^i C^{ij} \left(\frac{P^{ij}}{p^i}\right)^{\eta}$$

$$\text{Constraint} \quad \Rightarrow \quad P^{ij} C^{ij} = p^i C^i + p^j C^j$$

$$\text{CES Price} \quad \Rightarrow \quad P^{ij} = \left[\mu^i \left(p^i\right)^{1-\eta} + \mu^j \left(p^j\right)^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

Here the μ are scale parameters, and the η are of course elasticities.

2.10.4 Upper tree in the code

It is useful here to make the connection to the variable names and equations we observe in the code. Our five consumption goods have indices $cBil$ for cars, $cEne$ for energy, $cVar$ for goods, $cTur$ for tourism, $cTje$ for services. There is also an index entry $cBol$ for housing, although housing is not included in the tree. All these six individual indices are collected in an index set c.

The above sequence of CES cost minimization problems is only present in a compact way via intelligent indexing allowed by GAMS, so that a single instruction combining one expression for all first order conditions and another expression for all constraints solves the problem for the entire upper part of the tree.

Not only that, in the code we will not see a CES tree indexed by age. We assume the utility weights are identical across ages so that all cohorts have the same non-housing consumption decomposition which allows us to use total consumption $C_t = \sum_a C_{a,t} N_{a,t}$ in the tree problem.

$$p^C_{cNest,t} q^C_{cNest,t} = \sum_{\{c\_\}} q^C_{c\_,t} p^C_{c\_,t}$$

$$q^C_{c\_,t} = u^C_{c\_,t} q^C_{cNest,t} \left(\frac{p^C_{cNest,t}}{p^C_{c\_,t}}\right)^{e^C_{cNest}}$$

To make these expressions clearer we can look up at the figure of the upper tree. The object $q^C_{cNest,t}$ from the previous equations is any one of the nest objects $C^{CEGTS}_{a,t}$, $C^{EGTS}_{a,t}$, $C^{GTS}_{a,t}$, and $C^{TS}_{a,t}$, depending on which problem is being solved in these sets of equations.

The equality sign in these two equations is further controlled by a mapping cNest2c_[cNest,c_]. This mapping ensures the right branch of the tree is allocated to the right trunk object. The compact indexing relationship between the sets $cNest$ and $c\_$ is designed to include the entire upper tree. The set $cNest$ is a set with all upper nests in the consumption tree.[27] The set $c\_$ consists of all the components from both sets c and $cNest$. Notice also that the elasticity of substitution is indexed by $cNest$. This elasticity is not the same at all levels of the tree. Finally, in these equations $u^C_{c\_,t}$ are the scale parameters (utility weights μ) and e^C_{cNest} are elasticities (η).

[27] In the code $cNest = \{cX$, $cTurTjeVarEne$, $cTurTjeVar$, $cTurTje\}$ with cX being an index for the aggregate non durable consumption.

Table 1 below contains the values of these elasticities and also the *budget shares* of the different goods. The scale parameters follow budget shares in the data period (2017 and earlier) and in the forecast period (2018 and later) are given by ARIMA estimates. [28]

2.10.5 Lower tree in the code

From an organizational standpoint we do not need to include the two lower levels of the tree as a demand object. We can alternatively think of these two levels as a packaging intermediary that takes inputs from domestic and foreign production sources to produce a final consumption good.

Several simplifying assumptions make this notional decoupling of the lower tree from the household problem easier. First, the tree (including the upper tree) is the same for all ages. This implies we can work directly with demand aggregated over all ages (and we proceed below without the age index). Second, every problem in the tree is a zero profit object. This means we can take (for example) the outcome of the demand for cars from the upper tree in the household problem, and allocate it to the demand for output from the production sector using the mechanics of the lower tree without thinking of it as consumer behavior.

For these reasons the lower tree described here is also a key object in the input-output chapter where all aggregates are collected and the market clearing conditions are defined.

Lower tree, level 1, private production sources. In level 1 we source the five consumption goods from the nine production sectors. Here we assign fixed proportions, and we do so for all consumption goods. This is equivalent to having a Leontief demand and is the same structure used in the ADAM and SMEC models.

It is partly because we have a Leontief assignment in this level of the tree that we can think of the lower tree as technology rather than as consumer behavior. It is easier to think of technology as rigid than to think of an absolute inability to substitute between different consumption goods. However, because we have defined the problem of the production firm at the 9 sector decomposition, the structure that emanates from the 5 good consumption decomposition is also naturally a demand side object, and therefore we include this description here.

In terms of parameters, as we do not have an elasticity of substitution (it is zero), we have only the fixed proportions (scale parameters). For example we can see in Table 2 (2017 data) that for cars in level 1 we source them from manufacturing (circa 71%) and services (circa 29%) and so we have approximately

$$C_t^{Cars} = min\left(\frac{C_t^{Cars,man}}{\mu_{cars}^{man}}, \frac{C_t^{Cars,serv}}{\mu_{cars}^{serv}}\right) = min\left(\frac{C_t^{Cars,man}}{0.71}, \frac{C_t^{Cars,serv}}{0.29}\right)$$

so that equivalently

$$C_t^{Cars,man} = \mu_{cars}^{man} C_t^{Cars}$$

$$C_t^{Cars,serv} = \mu_{cars}^{serv} C_t^{Cars}$$

where generally we have $1 = \mu_{cars}^{serv} + \mu_{cars}^{man}$.

[28] "Estimering af Forbrugssystemet i MAKRO". Anders F. Kronborg og Christian S. Kastrup, March 2020.

For other consumption goods we have Leontief functions with different inputs. We can see in Table 2 (2017 data) that the consumption of the energy good uses mainly the production good energy ($\mu^{Ene}_{Energy} = 0.82$) and some produced services ($\mu^{Serv}_{Energy} = 0.16$), and a bit of manufacturing ($\mu^{man}_{Energy} = 0.02$). It does have other inputs of negligible contribution.

$$C_t^{Energy} = min\left(\frac{C_t^{Energy,Ene}}{\mu^{Ene}_{Energy}}, \frac{C_t^{Energy,Serv}}{\mu^{Serv}_{Energy}}, \frac{C_t^{Energy,man}}{\mu^{man}_{Energy}}\right)$$

In the code we have:

$$\frac{v^{IO}_{j,s,t}}{p^{IO}_{j,s,t}} = u^{IO}_{j,s,t} q^{J_j}_{j,t}, \quad j = \{r, c, k\}, J_j = \{R, C, I\}$$

where $v^{IO}_{j,s,t} = p^{IO}_{j,s,t} q^{IO}_{j,s,t}$. The upper index reads $J_r = R$, $J_c = C$, $J_k = I$. This system applies also to the demand by firms for intermediate inputs (r, R) and for investment goods (k, I), so all these lower tree constructions are contained in one equation in the code.

In this example the contribution of produced services to energy consumption has parameter

$$\mu^{Serv}_{Energy} = u^{IO}_{Energy,Services,t} = 0.16$$

Lower tree, level 1, public production sources. The above Leontief structure does not apply to private demand for public goods/services. This particular component, if and when present in any of the five consumption goods, is exogenized in the manner of the following hypothetical example.

Consider the consumption good "services" $C_t^{Services} \equiv N_{a,t} C_{a,t}^{Services}$. Remove from this total the quantity provided by the public sector, $C_t^{ServByGov}$. Then take the net services quantity, $C_t^{Services} - C_t^{ServByGov} = C_t^{NetS}$, and apply the Leontief structure from Table 2 (2017 data) to it:[29]

$$C_t^{NetS} = \left(1 - \mu^{Pub}_{Serv}\right) min\left(\frac{C_t^{NetS,man}}{\mu^{man}_{Serv}}, \frac{C_t^{NetS,serv}}{\mu^{serv}_{Serv}}, \frac{C_t^{NetS,sea}}{\mu^{sea}_{Serv}}\right) = 0.84 \times min\left(\frac{C_t^{NetS,man}}{0.02}, \frac{C_t^{NetS,serv}}{0.81}, \frac{C_t^{NetS,sea}}{0.01}\right)$$

Now, at this stage we would expect to have these coefficients sum to 1 and therefore filling the net services (net of public input) shares exactly:

$$\frac{0.02}{0.84} + \frac{0.81}{0.84} + \frac{0.01}{0.84} = 1$$

This is, however, not exactly true, although these factors do sum to extremely close to unity. The price of the "Leontief Output" in this case is given by the following equation:

$$P_t^{NetS} C_t^{NetS} = P_t^{man} C_t^{NetS,man} + P_t^{serv} C_t^{NetS,serv} + P_t^{sea} C_t^{NetS,sea}$$

where after substitution we obtain

$$P_t^{NetS} = P_t^{man} \frac{0.02}{0.84} + P_t^{serv} \frac{0.81}{0.84} + P_t^{sea} \frac{0.01}{0.84}$$

such that the "Leontief Output" price P_t^{NetS} is determined given the prices and coefficients, even if the sum of the μ is not exactly 1.[30]

[29] Note that we work with the aggregate quantities because the decomposition is the same for all ages. Furthermore, since all cohorts have the same tree structure, government services by age, $C_{a,t}^{ServByGov}$, are given by the total consumption by age relative to total consumption of all ages.

[30] Public sector contributions to private consumption of services follow public consumption (public

Table 2. For the entire consumption demand we have then 40 Leontief $\mu^{Poduction}_{Consumption}$ parameters after subtraction of the parameter/share of the demand for public goods. This share of public production affects only one consumption good, that of services. All parameters are collected in Table 2 below.[31] There we can see that in the first column "Pub" only the row describing services has a positive value.

Lower tree, level 2. In level 2 we source the subcomponents of the production part of our consumption good from domestic (dom) and foreign (for) sources, and we use a standard CES decomposition for that.[32] We have now scale parameters and elasticities. For the decomposition of the manufacturing subcomponent of cars, $C_t^{Cars,man}$, we have demand aimed at domestic sources, $C_t^{Cars,man,dom}$, given by the CES first order condition

$$C_t^{Cars,man,dom} = \mu^{man,dom}_{cars,t} C_t^{Cars,man} \left(\frac{P^{dom}_{man,t}}{P^{CES(dom,for)}_{cars,man,t}} \right)^{-\eta^{man}_{cars}}$$

and demand aimed at foreign sources, $C_t^{Cars,man,f}$, given by

$$C_t^{Cars,man,for} = \mu^{man,for}_{cars,t} C_t^{Cars,man} \left(\frac{P^{for}_{man,t}}{P^{CES(dom,for)}_{cars,man,t}} \right)^{-\eta^{man}_{cars}}$$

with scale parameters $\mu^{man,dom}_{cars,t}$ and $\mu^{man,for}_{cars,t}$ and elasticity η^{man}_{cars}. This elasticity is currently set at 1.25, and it is the same for all branches in the tree. This number is taken from the DREAM model. We are in the process of estimating different values for these parameters for the different branches.

The CES price solves the standard zero profit optimization problem and can be written directly

$$P^{CES(dom,for)}_{cars,man,t} = \left\{ \mu^{man,for}_{cars,t} \left(P^{for}_{man,t} \right)^{1-\eta^{man}_{cars}} + \mu^{man,dom}_{cars,t} \left(P^{dom}_{man,t} \right)^{1-\eta^{man}_{cars}} \right\}^{\frac{1}{1-\eta^{man}_{cars}}}$$

Given the prices which are exogenous to the consumer, and given the elasticities, the key assignment parameters that allocate demand are the scale parameters $\mu^{production,dom}_{demand,t}$, $\mu^{production,for}_{demand,t}$.

Other tables. Table 2 contains elasticities and budget shares in the upper tree. Budget shares are the corresponding fractions of nominal expenditure,

$$S^i = \frac{p^i_t q^i_t}{\sum_j p^j_t q^j_t}$$

Table 3 shows the Leontief proportionality factors in level 1 of the lower tree. Empty cells in Table 3 imply the consumption good of the respective row does not contain components from the production sector in the respective column.

expenditure G). For example, it is the part of kindergardens that is privately financed (the part you pay may be 5-10% of the real cost). These contributions are taken first and considered exogenous. μ^{Gov} is given by this exogenous amount. The other μ adjust so that the sum restriction is satisfied. Generally we would have $\sum \mu = 1$ and the sum restriction would be intuitive. However, in practice prices are not set to 1 in the base year.

[31] Again taken from "Estimering af Forbrugssystemet i MAKRO". Kronborg and Kastrup (2020).

[32] In the code domestic sources are labelled (y) as output, and foreign sources are labelled (m) as in imports.

Table 4 has information on the level 2 of the lower tree. Empty cells in Table 4 imply the consumption good in the respective row does not include goods produced in that column. They correspond to the empty cells in Table 3. Cells with a D imply there is production from that sector but only domestic production. Accordingly the foreign share is zero in the following row. Cells with an F imply there is only foreign supply from that sector into that consumption good and accordingly the foreign share in the following row will be one.

Table 2.1: Upper Tree Elasticities (η) and Budget Shares

	η	Budget Share 2010, 2017		
C and Housing	0.3	Cars	0.032	0.035
Cars $\rightarrow$ Nest	0.2	Energy	0.091	0.074
Energy $\rightarrow$ Nest	0.0	Goods	0.310	0.295
Goods $\rightarrow$ Nest	0.7	Services	0.325	0.342
Services and Tourism	1.1	Tourism	0.040	0.041
		Housing	0.202	0.213

Overall utility intertemporal elasticity of substitution is 1.
Budget shares are given by $S_i = p_i \times q_i / sum_j(p_j \times q_j)$.

Table 2.2: Lower Tree, Level 1. Leontieff Factors μ_{row}^{column}.

	Production Sectors, 2000 Data								
	Pub	Man	Agr	Ser	Ext	Con	Sea	Hou	Ene
Cars		0.58		0.42					
Energy		*	*	0.14					0.86
Goods		0.46	0.01	0.53	*	*	*	*	*
Services	0.18	0.01	*	0.80	*	*	0.01	*	*
Tourism				1.00					
	Production Sectors, 2017 Data								
	Pub	Man	Agr	Ser	Ext	Con	Sea	Hou	Ene
Cars		0.71		0.29					
Energy		0.02	*	0.16					0.82
Goods		0.43	0.01	0.56	*		*	*	*
Services	0.16	0.02	*	0.81	*	*	0.01	*	*
Tourism				1.00					

2000 and 2017 data. Rowsum = 1 (almost exactly). μ coefficients $\approx$ budget shares.
Empty cells => no input. Cells = * => Negligible input.

Table 2.3: Lower Tree Level 2. Elasticities η, and Foreign Share S_f.

		Production Sectors, 2017 Data								
		Pub	Man	Agr	Ser	Ext	Con	Sea	Hou	Ene
Cars	η		F		1.25					
	S_f		1		0.35					
Energy	η		1.25		1.25					1.25
	S_f		0.12		0.11					0.24
Goods	η		1.25	1.25	1.25					
	S_f		0.33	0.34	0.27					
Services	η	D	1.25		1.25			1.25		
	S_f	0	0.24		0.3			0.21		
Tourism	η				D					
	S_f				0					

Elasticities between foreign (f) and domestic (d) production. Tables entries are conditional on positive demand from the respective production sector. The foreign share is given by $S_f = p_f \times q_f/(p_f \times q_f + p_d \times q_d)$.

2.10.6 Tourism

There are both imports and exports of tourism. Imports of tourism consist of how much Danish households consume abroad and are given by the demand component $C_{'cTur',t}$ from the tree above. This is a normal consumption good and its demand increases with income. Exports of tourism are determined in the foreign sector chapter and its aggregate is given by $X_{'xTur',t}$. Total consumption of foreigners in Denmark is also divided into consumption groups in the foreign sector chapter and is given by $C^{Tourist}_{c,t}$.

The following object is useful in handling data. It is the value of consumption groups, $P^C_{c,t}C_{c,t}$, [33] which are given by the value of aggregate consumption of Danish households, $P^{CHH}_{c,t}C^{HH}_{c,t}$, and tourists, $P^{CTourist}_{c,t}C^{Tourist}_{c,t}$:[34]

$$P^C_{c,t}C_{c,t} = P^{CHH}_{c,t}C^{HH}_{c,t} + P^{CTourist}_{c,t}C^{Tourist}_{c,t}$$

Whereas in the model Danes and Tourists face the same prices for the same goods, in order to match the data they cannot face the same price for the same consumption components. We therefore use an adjustment factor

$$P^{CTourist}_{c,t} = \lambda^{pCTourist}_{c,t}P^{CHH}_{c,t}$$

where $\lambda^{pCTourist}_{c,t}$ is a parameter used to fit the data. It is assumed that this price margin remains constant going forward.

The value of aggregate Danish consumption does not include the consumption of foreign tourists in Denmark:

$$P^C_{'tot',t}C_{'tot',t} = \sum_c \left(P^C_{c,t}C_{c,t} - P^{CTourist}_{c,t}C^{Tourist}_{c,t}\right)$$

This implies $P^C_{'tot',t}C_{'tot',t} = P^{CHH}_{'tot',t}C^{HH}_{'tot',t}$. The quantities are almost identical, but there is a small difference because $C^{HH}_{'tot',t}$ is a CES-aggregate and $C_{'tot',t}$ is a chain-aggregate given by:

[33] This includes danes doing tourism and consuming in italy, and also italian tourists consuming in copenhagen. In fact, in the data the consumption of foreigners in Danmark is implicitly included in all consumption goods.

[34] This equation also applies for housing services which are not part of the tree. As we assume there is no tourist consumption of Danish housing this correction becomes zero.

$$P^{C}_{'tot',t-1} C_{'tot',t} = \sum_{c} \left(P^{C}_{c,t-1} C_{c,t} - P^{CTourist}_{c,t-1} C^{Tourist}_{c,t} \right)$$

2.11 Appendices - Households

2.11.1 Calculating the bequest allocation matrix

This section follows Boserup, Kopczuk, and Kreiner, (2016). Households take bequests received as exogenous and these enter the budget constraint as an additive term which is "hidden" inside the income variable. Even if agents receive bequests in the first period of economic life (age 18), we still have the initial condition for assets that $B_{0,t}$ is taken as given by the agent, as we exogenize transfers associated with children.

An individual of any given age receives bequests from agents deceased also at any given age. The distribution of bequests is modeled through a time varying matrix $M_t(a_d, a_h)$ where the indices refer respectively to the age of the deceased and to the age of the heir. This allocation matrix is general in that it encompasses all deaths, not just deaths of parents or grandparents, and all heirs.[35] As an example, children also die and leave assets to their parents and siblings.

When an individual dies in the model, he leaves a bequest. Assume that the individual dies at age a_d. The distribution matrix $M_t(a_d, a_h)$, describes the share of his bequest going to an average a_h year old individual. A given fraction of his wealth which he leaves as bequest is distributed equally by all agents of age a_h.

The distribution matrix. The matrix $M_t(a_d, a_h)$ is based on estimates of individual bequests from Danish administrative data. These estimates are obtained using a difference-in-difference estimator. This measures how the difference in wealth of an individual of age a_h, whose relative of age a_d has died, differs from the the difference in wealth of the average person of age a_h. This results in estimates of several specific bequests from a_d year old individuals to a_h year old individuals. Let i be the index for each specific transfer from an a_d year old to an a_h year old. $\widetilde{H}_{a_d,a_h,i,t}$ is then the estimated nominal amount transferred for each specific transfer. These bequests given by individuals in age group a_d to individuals in age group a_h are then summed and divided by the total number of a_d and a_h year olds (not just the ones involved in estimated bequest transfers but all individuals). The result is a data frame containing the average bequest $H_{a_d,a_h,t}$ received by an a_h year old from an a_d year old, regardless of whether a relative has died,

$$H_{a_d,a_h,t} = \frac{1}{N_{a_h,t} N_{a_d,t}} \left[\underbrace{\sum_i \widetilde{H}_{a_d,a_h,i,t}}_{\text{All transfers } a_d \text{to } a_h} \right]$$

where $N_{x,t}$ is the number of people of age group x. The age groups range from 0 to 100 and the time span is from 2000 to 2012.[36] The average bequest given/left by an individual from age group a_d is then given by

$$H_{a_d,t} = \sum_{a_h} H_{a_d,a_h,t} N_{a_h,t} = \frac{1}{N_{a_d,t}} \left[\underbrace{\sum_{a_h} \left(\sum_i \widetilde{H}_{a_d,a_h,i,t} \right)}_{\text{All transfers } a_d \text{to all } a_h} \right]$$

[35] The sample is larger than in Boserup, Kopczuk, and Kreiner, (2016). Nearly everyone who dies has a son or daughter, a parent, a nephew or niece, an uncle, etc. Therefore unaccounted would be only those who die completely alone, and yet have substantial assets to distribute. The odd bequest to a dog or cat may also fall outside our data.

[36] Note that age zero in the data corresponds to index 1 of age in the model.

Due to the sparsity of these matrices, they are averaged over time

$$H_{a_d} = \frac{1}{T}\sum_{t}^{T} H_{a_d,t}, \qquad H_{a_d,a_h} = \frac{1}{T}\sum_{t}^{T} H_{a_d,a_h,t}$$

The share of an a_d year old's bequests received by an a_h year old is then

$$\widetilde{\chi}_{a_d,a_h} = \frac{H_{a_d,a_h}}{H_{a_d}}$$

This share contains a large amount of noise. We therefore conduct a non-parametric estimation, using a local linear regression with the age of both giver and receiver as dependent variables and a Gaussian kernel. $\widetilde{\chi}_{a_d,a_h}$ is then replaced by the fitted value χ_{a_d,a_h}.
Since χ_{a_d,a_h} is time invariant, $\chi_{a_d,a_h} N_{a_h,t}$ will generally not sum to 1. This means that bequests given will not be the same as bequests received. To prevent this, the shares are normalized so that we finally obtain the allocation matrix

$$M_t\left(a_d, a_h\right) = \frac{\chi_{a_d,a_h}}{\sum_{a_h} \chi_{a_d,a_h} N_{a_h,t}}$$

These have the desired property that

$$\sum_{a_h} M_t\left(a_d, a_h\right) N_{a_h,t} = \frac{\sum_{a_h} \chi_{a_d,a_h} N_{a_h,t}}{\sum_{a_h} \chi_{a_d,a_h} N_{a_h,t}} = 1$$

Therefore total bequests given will equal total bequests received.

Consistency. Since people die at the end of a period the total bequest given by a deceased member of age group a_d, consists of his assets at the end of the period, which in the case of the model are net financial assets $B_{a_d,t}$ and housing.[37] Here we proceed using only net financial assets B as an illustration. This means that the average bequest given by a member of age group a_d is $\left(1 - s_{a_d,t}\right) B_{a_d,t}$ where $s_{a_d,t}$ is the survival rate, i.e. the probability of an a_d year old also being alive at age $a_d + 1$. Total bequests given by all age groups at the end of time t after all decisions have been taken are then

$$H_t = \sum_{a_d} \left(1 - s_{a_d,t}\right) B_{a_d,t} N_{a_d,t}$$

Bequests are received in the next period. The average bequest from an a_d year old deceased at the end of period $t-1$ received by an a_h year old in period t will therefore be

$$\left(1 - s_{a_d,t-1}\right) B_{a_d,t-1} M_t\left(a_d, a_h\right)$$

The bequest received by a member of age group a_h at time t is then given by

$$H_{a_h,t} = \sum_{a_d} \left(1 - s_{a_d,t-1}\right) B_{a_d,t-1} M_t\left(a_d, a_h\right) N_{a_d,t-1}$$

[37] In the data the value of property is included in the wealth difference such that the allocation matrix we use is consistent with the model.

This in turn results in total bequests received being

$$\begin{aligned}
\sum_{a_h} H_{a_h,t} N_{a_h,t} &= \sum_{a_h} \left(\sum_{a_d} \left(1 - s_{a_d,t-1}\right) B_{a_d,t-1} M_t \left(a_d, a_h\right) N_{a_d,t-1} \right) N_{a_h,t} \\
&= \sum_{a_h} \left(\sum_{a_d} \left(1 - s_{a_d,t-1}\right) B_{a_d,t-1} \frac{\chi_{a_d,a_h}}{\sum_{a_h} \chi_{a_d,a_h} N_{a_h,t}} N_{a_d,t-1} \right) N_{a_h,t} \\
&= \sum_{a_d} \left(1 - s_{a_d,t-1}\right) B_{a_d,t-1} \left(\sum_{a_h} \frac{\chi_{a_d,a_h} N_{a_h,t}}{\sum_{a_h} \chi_{a_d,a_h} N_{a_h,t}} \right) N_{a_d,t-1} \\
&= \sum_{a_d} \left(1 - s_{a_d,t-1}\right) B_{a_d,t-1} N_{a_d,t-1} = H_{t-1}
\end{aligned}$$

so that total bequests given last period equal total bequests received this period[38].

2.11.2 Land and housing depreciation

The housing $D_{a,t}$ the agent owns is an aggregate object containing "bricks" and land. The entire stock of land is held by households inside their housing good. An intermediary buys "bricks" and buys land released from depreciated housing, packages these together and sells the resulting housing good to families. Here we make an important simplification to the model for practical reasons. As over time the exact composition of new housing in terms of bricks and land may change, so does the implicit composition in terms of bricks and land of the total housing holdings, and this affects households of different ages differently. We simplify the model by assuming that the composition of housing in terms of bricks and land is always identical for all households. This avoids having to trace two additional age specific stock variables (bricks and land) inside the household problem, and is similar to the assumption used in the labor market where the age distribution of workers is the same in every firm.

Now, inside the housing good "bricks" depreciate but land does not. Nevertheless, the depreciation rate of the housing object is still the depreciation rate of bricks, as the land associated with depreciated bricks is released and sold by the household. Therefore we account for the released land as "lost" in the normal law of motion

$$z_{a,t} = D_{a,t} - \left(1 - \delta_t^{bricks}\right) D_{a-1,t-1}$$

and "recover" it as household revenues from land sales.

One final detail is that new land is released into the economy every period. The aggregate land variable grows exogenously and this land growth is helicopter dropped on households proportionally to their individual land holdings. In order to settle the accounting of land sales we must determine the individual land holdings relative to aggregate land.

Unconstrained agents own the following fraction of total land:

$$\left(\frac{D_{a-1,t-1}^{unc} \times (1 - \Upsilon) N_{a-1,t-1}}{\sum_a \left(\Upsilon D_{a-1,t-1}^{cons} + (1 - \Upsilon) D_{a-1,t-1}^{unc} \right) N_{a-1,t-1}} \right) \frac{1}{(1 - \Upsilon) N_{a-1,t-1}}$$

The term in squared brackets contains the fraction of total land held by the cohort. The second term is 1 over the cohort size. The product of the two yields the fraction of

[38] The amount actually available as disposable income for the receiver differs from the amount given due to transaction costs, taxes and interests payments.

individual land holdings. Eliminating terms this equals

$$D^{unc}_{a-1,t-1}\left(\frac{1}{\sum_a\left(\Upsilon D^{cons}_{a-1,t-1}+(1-\Upsilon)\,D^{unc}_{a-1,t-1}\right)N_{a-1,t-1}}\right)\equiv D^{unc}_{a-1,t-1}\Omega^{Land}_t$$

Using the same reasoning unconstrained agents have the following fraction of total land: $D^{cons}_{a-1,t-1}\Omega^{Land}_t$. The term Ω^{Land}_t is of course the same for both types.

Now we are ready to determine revenues from land sales. The total quantity of land being sold is the land released by housing depreciation plus the helicopter land growth, $Land^{sales}_t=\delta^{bricks}_t Land_{t-1}+Land_t-Land_{t-1}$. Individual revenues from selling land are then given by

$$D_{a-1,t-1}\Omega^{Land}_t P^{Land}_t Land^{Sales}_t$$

This quantity is now adapted to the model in the main text by defining the object α^{Land}_t. This is given by

$$\alpha^{Land}_t=\frac{\Omega^{Land}_t P^{Land}_t Land^{Sales}_t}{P^D_{t-1}}$$

where α^{Land}_t is the same for all types and ages.

A final remark regarding depreciation is in order. Housing depreciation can be endogenous. Maintenance investments prolong the life of a house. Such investments amount to home production or to purchases from small to medium size service providers such as plummers and carpenters. This can be modelled by extending the law of motion into

$$D_{a,t}=\left(1-\delta_t+\hat{\delta}\left(y^m_t\right)\right)D_{a-1,t-1}+z_{a,t}$$

and adding an expenditure item y^m_t in the budget constraint. This level of detail can be implemented later if necessary.

2.11.3 Utility function, rigidity, and reference consumption

Utility. The unconstrained household maximizes the present discounted value of utility flows. The present value of this sequence must account for the possibility of death along the way. Denoting the utility of consumption as U and the utility of bequests as W this sequence can be pictured as follows

$$\begin{tikzcd} U_{a,t} \arrow[r, dash] \arrow[dr, dash] & \beta s_{t+1}U_{t+1} \arrow[r, dash] \arrow[dr, dash] & \beta s_{t+2}U_{t+2} \\ & \beta\left(1-s_{t+1}\right)W_t & \beta\left(1-s_{t+2}\right)W_{t+1} \end{tikzcd}$$

In game theory language, death is an exit from the game tree. Every item in the game tree has a respective probability calculated from the perspective of an agent who is alive at time t. These probabilities change as life expectancy evolves over time. This sequence then has a summation representation $S_{a,t}$ (now with age and time indices)

$$S_{a,t}=U_{a,t}+\sum_{j=1}^{A}\left(\prod_{i=1}^{j}\beta_{a+i,t+i}\right)\left(\prod_{i=1}^{j-1}s_{a+i,t+i}\right)\left[s_{a+j,t+j}U_{a+j,t+j}+\left(1-s_{a+j,t+j}\right)W_{a+j-1,t+j-1}\right]$$

It is this object that our optimizing agent in MAKRO optimizes each period.

CES utility flow. A large number of references in the literature use a Cobb-Douglas specification, but many use also the CES function which is the one we use.

$$U_{a,t} = \frac{1}{1-\eta}\left[\tilde{U}_{a,t}\right]^{1-\eta}$$

$$\tilde{U}_{a,t} \equiv \left[\left(\upsilon^{c}_{a,t}\right)^{\frac{1}{E}}\left(C_{a,t}\right)^{\frac{E-1}{E}} + \left(\upsilon^{d}_{a,t}\right)^{\frac{1}{E}}\left(D_{a,t}\right)^{\frac{E-1}{E}}\right]^{\frac{E}{E-1}}$$

with derivatives

$$\frac{\partial U_{a,t}}{\partial C_{a,t}} \equiv U^{1}_{a,t} = \left[\tilde{U}_{a,t}\right]^{-\eta} \times \left(\frac{\upsilon^{c}_{a,t}\tilde{U}_{a,t}}{C_{a,t}}\right)^{\frac{1}{E}}$$

and

$$\frac{\partial U_{a,t}}{\partial D_{a,t}} \equiv U^{2}_{a,t} = \left[\tilde{U}_{a,t}\right]^{-\eta}\left(\frac{\upsilon^{d}_{a,t}\tilde{U}_{a,t}}{D_{a,t}}\right)^{\frac{1}{E}}$$

Reference consumption and household size. We use a reference target for consumption and housing to calibrate rigidity. We write

$$U_{a,t} \equiv U\left(\tilde{C}_{a,t}, \tilde{D}_{a,t}\right)$$

Here $\tilde{C}_{a,t}$ denotes consumption net of a reference quantity with a coefficient χ:

$$\tilde{C}_{a,t} = \frac{C_{a,t}}{\zeta_{a,t}} - \chi^{C}\frac{C_{a-1,t-1}}{\zeta_{a-1,t-1}}$$

The weight, $\zeta_{a,t}$, depends on the number of children in the household[39]

$$\zeta_{a,t} = 1 + \frac{1}{2}n^{\text{children}}_{a,t}$$

We do the same for housing by considering the following object inside utility

$$\tilde{D}_{a,t} = \frac{D_{a,t}}{\zeta_{a,t}} - \chi^{D}\frac{D_{a-1,t-1}}{\zeta_{a-1,t-1}}$$

The reference quantities $C_{a-1,t-1}$ and $D_{a-1,t-1}$ can be viewed as the average of the cohort in the previous period, rather than the individual household's own previous decisions. In this way they are exogenous to the household.

Total cohort consumption, $C^{total}_{a,t}$, is given by the sum of the consumption of rational and irrational agents

$$C^{total}_{a,t} = N_{a,t}\left[(1-\Upsilon)C^{unc}_{a,t} + \Upsilon C^{con}_{a,t}\right]$$

and likewise for the housing stock

$$D^{total}_{a,t} = N_{a,t}\left[(1-\Upsilon)D^{unc}_{a,t} + \Upsilon D^{con}_{a,t}\right]$$

[39]Children are a late addition to this model. They are a noticeable life cycle pattern which affects household savings and consumption behaviour and it is useful to make it explicit. The object $\zeta_{a,t}$ requires changing the way the first order conditions are writen in the text but the change is marginal and therefore we do not make it explicit in the text at this stage. The contribution of this variable seems also to be largely caught in the CES utility parameters $\upsilon^{j}_{a,t}$ we recover, and therefore this addition to the model is currently under review.

2.11.4 Mortgages and Housing in the budget constraint

Preliminaries. The law of motion for the housing stock is

$$D_{a,t} = \left(1 - \delta^d\right) D_{a-1,t-1} + z_{a,t}$$

When we derive the budget constraint we consider the cases of positive versus negative net investment in housing since when $z_{a,t} > 0$ we want to to impose a down-payment constraint but when $z_{a,t} < 0$ we do not.

In order to make the budget constraint below easier to read define the composite variable

$$\Delta_{a,t} \equiv B_{a,t} - \left(1 + r^h_{a,t}\right) B_{a-1,t-1} - yDisp_{a,t} + rent_t H_{a,t}$$

We postulate the exogenous relationship for the mortgage debt stock $X^M_{a,t}$, such that mortgages are proportional to the value of the house

$$X^M_{a,t} = \mu_{a,t} P^D_t D_{a,t}$$

where $\mu_{a,t}$ is a variable which is exogenous to the household, and which we detail below. Endogenous mortgage ratios would not only add choices and variables to the problem, but also imply handling corner solutions which would be computationally problematic given the size of the model.

The budget constraint: positive investment in housing. Consider first the case of $z_{a,t} > 0$. The term $M^{DP}_{a,t} > 0$ is the fraction or amount paid in cash when increasing the housing stock (the down-payment), and $m_{a,t}$ is an unspecified mortgage payment. In this case the size of the mortgage stock obeys the law of motion

$$X^M_{a,t} = \left(1 + r^{mort}_t\right) X^M_{a-1,t-1} + P^D_t Z_{a,t} - M^{DP}_{a,t} - m_{a,t}$$

The budget constraint of the household is

$$\begin{aligned}\Delta_{a,t} + P^C_t C_{a,t} &= -M^{DP}_{a,t} - m_{a,t} \\ &- \left(\tau^W_t + x_t\right) P^D_{t-1} D_{a-1,t-1} + P^D_{t-1} D_{a-1,t-1} \alpha^{Land}_t\end{aligned}$$

where τ^W is the wealth tax rate, x_t measures expenses in running the property, and the last term is the revenue from land sales. Now use $X_{a,t} = \mu_{a,t} P^D_t D_{a,t}$, and the laws of motion for D and X^M to get

$$\begin{aligned}\Delta_{a,t} + P^C_t C_{a,t} = &- \left(1 + r^{mort}_t\right) \mu_{a-1,t-1} P^D_{t-1} D_{a-1,t-1} + \mu_{a,t} P^D_t D_{a,t} \\ &- P^D_t D_{a,t} + P^D_t \left(1 - \delta_t\right) D_{a-1,t-1} \\ &- \left(\tau^W_t + x_t\right) P^D_{t-1} D_{a-1,t-1} + P^D_{t-1} D_{a-1,t-1} \alpha^{Land}_t\end{aligned}$$

The budget constraint: negative investment in housing. Consider now the case of $z_{a,t} < 0$. The budget constraint of the household does not have a down payment fraction but rather keeps the entire proceeds of the net sale

$$\begin{aligned}\Delta_{a,t} + P^C_t C_{a,t} &= -P^D_t Z_{a,t} - m_{a,t} \\ &- \left(\tau^W_t + x_t\right) P^D_{t-1} D_{a-1,t-1} + P^D_{t-1} D_{a-1,t-1} \alpha^{Land}_t\end{aligned}$$

Since none of the revenues are used to pay down the mortgage, the size of the mortgage stock obeys the law of motion

$$X^M_{a,t} = \left(1 + r^{mort}_t\right) \times X^M_{a-1,t-1} - m_{a,t}$$

When we put the two together we obtain exactly as above

$$\Delta_{a,t} + P_t^C C_{a,t} = -\left(1 + r_t^{mort}\right) \mu_{a-1,t-1} P_{t-1}^D D_{a-1,t-1} + \mu_{a,t} P_t^D D_{a,t}$$
$$-P_t^D D_{a,t} + P_t^D \left(1 - \delta_t\right) D_{a-1,t-1}$$
$$-\left(\tau_t^W + x_t\right) P_{t-1}^D D_{a-1,t-1} + P_{t-1}^D D_{a-1,t-1} \alpha_t^{Land}$$

There is no asymmetry in the problem. This makes sense. Once you fix exogenously the mortgage ratio, it does not matter whether net investment is positive or negative since the size of the mortgage is no longer a choice. Of note is also the fact that the mortgage payment $m_{a,t}$ disappears entirely from the problem.

The f object. Reorganizing terms yields the f object we use in the main text.

$$f\left(D_{a,t}, D_{a-1,t-1}\right) = \left(1 - \mu_{a,t}\right) P_t^D D_{a,t}$$
$$+\left\{\left(1 + r_t^{mort}\right) \mu_{a-1,t-1} + \tau_t^W + x_t - \frac{P_t^D}{P_{t-1}^D}\left(1 - \delta_t^d\right) - \alpha_t^{Land}\right\} P_{t-1}^D D_{a-1,t-1}$$

The Mortgage Ratio μ. Mortgages are proportional to the value of the house

$$X_{a,t}^M = \mu_{a,t} P_t^D D_{a,t}$$

where $\mu_{a,t}$ is exogenous to the household and is given by

$$\mu_{a,t} = \tilde{\mu}_{a,t} \frac{\bar{P}_{a,t}^D}{P_t^D}$$

where $\tilde{\mu}_{a,t}$ is a calibration object exogenous to the model.

The reference price $\bar{P}_{a,t}^D$ is a function of current and past prices of the form

$$\bar{P}_{a,t}^D = \Gamma_{a,t} P_t^D + \left(1 - \Gamma_{a,t}\right) \bar{P}_{a-1,t-1}^D$$

The factor $\Gamma_{a,t}$ is a measure of the number of new mortgages. A simple measure is the ratio of current investment over final stock

$$\Gamma_{a,t} = \frac{Z_{a,t}}{D_{a,t}} = \frac{D_{a,t} - \left(1 - \delta^d\right) D_{a-1,t-1}}{D_{a,t}}$$

In this way, for the first age of economic life when houses are bought, $\Gamma_{a,t}$ will be 1 implying all mortgages are new and subject to the current price. This number $\Gamma_{a,t}$ is bounded above by 1 and since D is always positive it has a finite lower bound. Younger agents are much more subject to the variation in house prices than older ones.[40] The ability to finance through a mortgage therefore varies with house prices. Given $\Gamma_{a,t}$ the ratio

$$\mu_{a,t} = \tilde{\mu}_{a,t} \frac{\bar{P}_{a,t}^D}{P_t^D} = \tilde{\mu}_{a,t} \left(\Gamma_{a,t} + \left(1 - \Gamma_{a,t}\right) \frac{\bar{P}_{a-1,t-1}^D}{P_t^D}\right)$$

falls at impact with an increase in house prices. The household can mortgage more as prices increase, since $X_{a,t}^M = \mu_{a,t} P_t^D D_{a,t}$ increases with the house price keeping all else constant, but less than proportionally. Leverage ratios fall with house price increases. But

[40] We can of course use a value of Γ that is constant over the life cycle. Our model of firm debt makes it proportional to the firm's capital stock. There, although we do not do so at this moment, we can use this exact specification with a constant Γ as the firm has no life cycle.

since $\Gamma_{a,t}$ changes endogenously as housing decisions react to house prices the mortgage ratio is more reactive. Leverage ratios fall slightly more with house price increases if investment falls. The exact effect depends on how persistent the increase in prices is, which affects the investment decision. A temporary increase in house prices should trigger a strong fall in investment. A permanent one not necessarily so. We use a slightly more general way of writing this factor as follows

$$\Gamma_{a,t} = (1-\phi) + \phi \frac{D_{a,t} - \left(1-\delta^d\right) D_{a-1,t-1}}{D_{a,t}} = 1 - \phi \left(1-\delta^d\right) \frac{D_{a-1,t-1}}{D_{a,t}}$$

as it allows for a degree of control over the influence of the endogenous housing decision on the mortgage ratios. The object $\phi\left(1-\delta^d\right)$ appears as a single constant in the model code.

No transaction costs. Proper aggregation of non convexities at the micro level, such as fixed costs of trading houses, is necessary for an accurate description of aggregate behavior.[41] Given the constraints imposed by the GAMS software and by the size of the model introducing these non differentiabilities excessively increases the computational burden. In the absence of an endogenous trade-off between renting and owning we leave such adjustment costs out of the problem, and proxy for them through the reference housing value inserted into the utility function.

[41]See the entire literature on firm investment with non convexities. Specific examples are Cooper and Adda (2000) on cars, Li, Liu, Yang, and Yao (2016) on housing, and Ampudia, Cooper, LeBlanc and Zhu (2019) on financial portfolio adjustment.

2.11.5 Aggregation

Due to migration flows, population obeys

$$N_{a,t} = s_{a-1,t-1}N_{a-1,t-1} + I_{a,t} - E_{a,t}$$

and while in the data it is clear that immigrants and emigrants are different from the average household in most respects, the model is nevertheless bound by the necessity to fit all agents into an average that can be replicated.[42] The household model has two dimensions of heterogeneity. One is age, and the other is the presence of HTM agents. Any additional heterogeneity is eliminated.

The goal is then to generate average quantities of assets B, housing D, consumption C, and employment that encompass residents and migrants in an internally consistent way. In the labor market chapter we detail the assumptions and mechanics needed to generate average employment, and here we detail the aggregation of assets and housing. We therefore assume that migrants carry with them the necessary assets to appropriately fit the resulting average. That is not the only assumption required, so we first work through an aggregation example without housing, and then replicate it with added housing.

Aggregation without housing. We **first** detail the budget constraint of three groups of households. Surviving residents $s_{a-1,t-1}N_{a-1,t-1} - E_{a,t}$ have the budget constraint where $x_{a,t} \equiv y_{a,t} - P_t^C C_{a,t}$ and where we have added a transfer term $T_{a,t}^S$ (superscript S for stayers):

$$B_{a,t} - x_{a,t} = (1 + r_{a,t})\, B_{a-1,t-1} + T_{a,t}^S$$

The people who leave, $E_{a,t}$ receive/pay a transfer $T_{a,t}^E$ before leaving. They do not earn income, consume or buy end-of-period assets in the country this period so that their constraint is simply a definition of the amount $M_{a,t}$ they carry abroad:

$$M_{a,t} \equiv (1 + r_{a,t})\, B_{a-1,t-1} + T_{a,t}^E$$

Finally, the people who enter the country, $I_{a,t}$ receive/pay a transfer $T_{a,t}^I$ after arrival. This transfer and the assets A they bring into the country allow them to make identical choices to those of the first group (the stayers):

$$B_{a,t} - x_{a,t} = A_{a-1,t-1} + T_{a,t}^I$$

The **next** step is to define some properties we need to impose. We choose to impose a zero net effect on the balance of payments.[43] This implies the following relationship between households entering and exiting the country:

$$A_{a-1,t-1} = \frac{E_{a,t}}{I_{a,t}} M_{a,t} = \frac{E_{a,t}}{I_{a,t}} (1 + r_{a,t})\, B_{a-1,t-1} + \frac{E_{a,t}}{I_{a,t}} T_{a,t}^E$$

We then impose that the transfer flows have zero net sum. This implies:

$$[s_{a-1,t-1}N_{a-1,t-1} - E_{a,t}]\, T_{a,t}^S + E_{a,t}T_{a,t}^E + I_{a,t}T_{a,t}^I = 0$$

and finally we impose equality of stayers and new arrivals which implies

$$T_{a,t}^I = (1 + r_{a,t})\, B_{a-1,t-1} + T_{a,t}^S - A_{a-1,t-1}$$

[42] Non migrants are also heterogeneous and yet only their average by age is in the model.

[43] This can be changed.

Now we can aggregate to obtain the budget constraint of the population $N_{a,t}$. This is done in the following steps. Add surviving residents $s_{a-1,t-1}N_{a-1,t-1} - E_{a,t}$ plus incomers

$$N_{a,t}\left[B_{a,t} - x_{a,t}\right] = \left[s_{a-1,t-1}N_{a-1,t-1} - E_{a,t}\right]\left[\left(1 + r_{a,t}\right)B_{a-1,t-1} + T^S_{a,t}\right] + I_{a,t}\left[A_{a-1,t-1} + T^I_{a,t}\right]$$

replace the zero balance of payments relationship

$$N_{a,t}\left[B_{a,t} - x_{a,t}\right] = \left[s_{a-1,t-1}N_{a-1,t-1} - E_{a,t}\right]\left[\left(1 + r_{a,t}\right)B_{a-1,t-1} + T^S_{a,t}\right]$$
$$+ I_{a,t}\left[\frac{E_{a,t}}{I_{a,t}}\left(1 + r_{a,t}\right)B_{a-1,t-1} + \frac{E_{a,t}}{I_{a,t}}T^E_{a,t} + T^I_{a,t}\right]$$

replace T^I with the equality of stayers and incomers and collect terms to obtain

$$B_{a,t} - x_{a,t} = \left(1 + r_{a,t}\right)B_{a-1,t-1} + T^S_{a,t}$$

Finally, determine the transfer payments T. If we put together the relationshipson zero balance of payments, zero net transfers, and the equality of stayers and new arrivals, we obtain that

$$T^S_{a,t} = -\frac{\left(I_{a,t} - E_{a,t}\right)}{N_{a,t}}\left(1 + r_{a,t}\right)B_{a-1,t-1}$$
$$T^I_{a,t} = \left(I_{a,t} - E_{a,t}\right)\left[\frac{1}{I_{a,t}} - \frac{1}{N_{a,t}}\right]\left(1 + r_{a,t}\right)B_{a-1,t-1} - \frac{E_{a,t}}{I_{a,t}}T^E_{a,t}$$

and using this we can rewrite the aggregate budget constraint as

$$B_{a,t} - x_{a,t} = \left(1 + r_{a,t}\right)B_{a-1,t-1}\Gamma_{a,t}$$

where

$$\Gamma_{a,t} = \frac{s_{a-1,t-1}N_{a-1,t-1}}{N_{a,t}}$$

One last detail is that the transfer $T^E_{a,t}$ is arbitrary and can be set at zero. These transfers are artificial constructions used to ensure new arrivals and surviving residents can make the same exact decisions subject to a balance of payments constraint which in this case is zero. They are added to the problem as a lump sum which is taken as exogenous by the household and do not affect marginal decisions. Furthermore, as all agents make identical choices, the only budget constraint that needs to be satisfied is the aggregate one, namely

$$B_{a,t} - x_{a,t} = \left(1 + r_{a,t}\right)B_{a-1,t-1}\Gamma_{a,t}$$

Aggregation with housing. Surviving residents $s_{a-1,t-1}N_{a-1,t-1} - E_{a,t}$ have the budget constraint

$$B_{a,t} - x_{a,t} + \left(1 - \mu_{a,t}\right)P^D_t D_{a,t} = \left(1 + r_{a,t}\right)B_{a-1,t-1} - \chi_{a,t}P^D_{t-1}D_{a-1,t-1} + T^S_{a,t}$$

where $x_{a,t} = y_{a,t} - P^C_t C_{a,t}$ and

$$\chi_{a,t} \equiv \left(1 + r^{mort}_t\right)\mu_{a-1,t-1} + \tau^W_t + x_t - \frac{P^D_t}{P^D_{t-1}}\left(1 - \delta^d_t\right) - \alpha^{Land}_t$$

The people who leave, $E_{a,t}$ do not consume or work in the country, and sell their houses so that $z_{a,t} = -\left(1 - \delta_t\right)D_{a-1,t-1} < 0$ and take their assets and proceedings abroad. Because they are downsizing, their budget constraint does not have a down payment fraction but rather keeps the entire proceeds of the sale. The final mortgage payment m liquidates the outstanding mortgage so that we obtain

$$M_{a,t} \equiv (1 + r_{a,t}) B_{a-1,t-1} - \chi_{a,t} P^D_{t-1} D_{a-1,t-1} + T^E_{a,t}$$

and where $M_{a,t}$ is the amount these agents will take abroad.

The people who enter the country, $I_{a,t}$ have $z_{a,t} = D_{a,t} > 0$ and they earn their income and consume here while they bring assets $A_{a-1,t-1}$ from abroad. Their housing expenditure is $(1 - \mu_a) P^D_t D_{a,t}$. Their full budget constraint is

$$B_{a,t} - x_{a,t} + (1 - \mu_{a,t}) P^D_t D_{a,t} = A_{a-1,t-1} + T^I_{a,t}$$

We now impose the three conditions from above. Once again the zero balance of payments effect is defined as

$$A_{a-1,t-1} = \frac{E_{a,t}}{I_{a,t}} M_{a,t}$$

and the property that the transfer flows have zero net sum is also as above:

$$[s_{a-1,t-1} N_{a-1,t-1} - E_{a,t}] T^S_{a,t} + E_{a,t} T^E_{a,t} + I_{a,t} T^I_{a,t} = 0$$

and finally the equality of stayers and new arrivals now implies

$$T^I_{a,t} = (1 + r_{a,t}) B_{a-1,t-1} - \chi_{a,t} P^D_{t-1} D_{a-1,t-1} + T^S_{a,t} - A_{a-1,t-1}$$

Reproducing the same steps performed above we will find that we can set $T^E_{a,t} = 0$ and that we obtain the following objects. The aggregate budget constraint

$$B_{a,t} + (1 - \mu_{a,t}) P^D_t D_{a,t} = y_{a,t} - P^C_t C_{a,t}$$

$$+ \left[(1 + r_{a,t}) B_{a-1,t-1} - \chi_{a,t} P^D_{t-1} D_{a-1,t-1}\right] \Gamma_{a,t}$$

and the transfer

$$T^S_{a,t} = -\frac{(I_{a,t} - E_{a,t})}{N_{a,t}} \left[(1 + r_{a,t}) B_{a-1,t-1} - \chi_{a,t} P^D_{t-1} D_{a-1,t-1}\right]$$

and again, as all agents make identical choices, the only budget constraint that needs to be satisfied is the aggregate one.

2.11.6 Housing intermediary

Households buy houses from a particular intermediary agent. The purpose of having this agent is to introduce land in the model. Although studies for other countries are only indicative of possible effects in Danmark, they document land as a fundamental factor affecting house prices.[44] The housing intermediary buys bricks from the construction sector and land from households, packages them and sells them to households as houses. These are the new houses q_t^n built in a given period and correspond to the aggregate net investment obtained in the household problem after summing over age, $q_t^n \equiv D_t - (1-\delta_t) D_{t-1}$.

The intermediary agent has profits given by

$$\pi_t = P_t^D q_t^n - P_t^{Land} q_t^{LS} - P_t^I q_t^I$$

where new houses q_t^n equal gross output minus adjustment costs, $q_t^n = q_t^y - q_t^{ac}$, and gross output q_t^y uses land q_t^{LS} and "bricks" q_t^I which here are labelled as an investment quantity because they will be used to accumulate an auxiliary stock of buildings needed to match data. The adjustment cost function is added to the problem here in order to help fit dynamic behaviour of house prices and quantities.[45] This function is given by

$$q_t^{ac} = \frac{\gamma}{2} q_{t-1}^I \left(\frac{q_t^I}{q_{t-1}^I} - \xi_t^I \right)^2$$

while the first order conditions are

$$P_t^D \left[\frac{\partial q_t^y}{\partial q_t^{LS}} \right] \equiv P_t^D \left(\mu_{LS} \frac{q_t^y}{q_t^{LS}} \right)^{1/E} = P_t^{Land}$$

$$P_t^D \left[\frac{\partial q_t^y}{\partial q_t^I} - \frac{\partial q_t^{ac}}{\partial q_t^I} \right] = P_t^I + \beta_{t+1} P_{t+1}^D \left[\frac{\partial q_{t+1}^{ac}}{\partial q_t^I} \right]$$

It is useful to detail the second condition which is dynamic and forward looking due to the adjustment cost function

$$P_t^D \left[\left(\mu_I \frac{q_t^y}{q_t^I} \right)^{1/E} - \gamma \left(\frac{q_t^I}{q_{t-1}^I} - \xi_t^I \right) \right] = P_t^I + \beta_{t+1} P_{t+1}^D \left[\frac{\gamma}{2} \left(\frac{q_{t+1}^I}{q_t^I} \right)^2 + \frac{\gamma}{2} \left(\xi_{t+1}^I \right)^2 \right]$$

so that we can write demand functions in CES style

$$q_t^{LS} = \mu_{LS} \cdot q_t^y \left(\frac{P_t^{Land}}{P_t^D} \right)^{-E}$$

$$q_t^I = \mu_I \cdot q_t^y \left(\frac{user_t}{P_t^D} \right)^{-E}$$

$$user_t = P_t^I + \gamma P_t^D \left\{ \frac{q_t^I}{q_{t-1}^I} - \xi_t^I + \beta_{t+1} \frac{P_{t+1}^D}{P_t^D} \frac{1}{2} \left[\left(\frac{q_{t+1}^I}{q_t^I} \right)^2 + \left(\xi_{t+1}^I \right)^2 \right] \right\}$$

[44] Davis, M. A., and Heathcote, J. (2005). Housing and the Business Cycle. International Economic Review, Vol. 46, No. 3, pp. 751-784. Davis, M. A. and Heathcote, J. (2007). The price and quantity of residential land in the United States. Journal of Monetary Economics, vol. 54(8), pp. 2595-2620.

[45] Ideally these adjustment costs would be introduced at the individual household level but that increased significantly the computational burden due to the large life cycle. Also, at the age specific household level they would mostly be fixed costs, something even more computationally demanding.

Given these demand function the optimization problem solves using both of them and the CES zero profit condition

$$P_t^D q_t^y = P_t^{Land} q_t^{LS} + user_t q_t^I$$

The quantity q_t^I is then used to generate the stock variable q_t^k in the standard manner

$$q_t^k = (1-\delta_t)\, q_{t-1}^k + q_t^I$$

where δ is the depreciation rate of the "bricks" or buildings part of the house. This is a measure of the stock of construction in owned housing, the total stock of "bricks" contained in all existing owned housing. To this we add an exogenous capital stock of rental housing to get the stock variable that enters the housing production sector which is present in the input-output data, and which is described in the chapter on private production.

Merging profits of intermediary with the household budget constraint. Due to the addition of adjustment costs these intermediary agents have profits. As this is an auxiliary construction to introduce land into the model which does not have a correspondence in data, these profits are plugged back into the household budget constraint as transfers which do not affect the user cost and the marginal decisions of households.[46] These profits are allocated to households by age using the weight $D_{a-1,t-1}/D_{t-1}$ as illustrated in the optimizing household budget constraint.

$$\begin{aligned}
B_{a,t} &= B_{a-1,t-1} + r_{a,t} B_{a-1,t-1} + \tilde{y}_{a,t} - p_t^c c_{a,t} - (1-\mu_{a,t})\, P_t^D D_{a,t} \\
&- \left\{ \left(1 + r_t^{mort}\right) \mu_{a-1,t-1} + \tau_t^W + x_t - \frac{P_t^D}{P_{t-1}^D}\left(1-\delta_t^d\right) - \alpha_t^{Land} \right\} P_{t-1}^D D_{a-1,t-1} \\
&+ \underbrace{\frac{D_{a-1,t-1}}{D_{t-1}}}_{weight} \left\{ \underbrace{P_t^D \left[D_t - \left(1-\delta_t^d\right) D_{t-1}\right] - P_t^{Land} q_t^{LS} - P_t^I q_t^I}_{profits} \right\}
\end{aligned}$$

At this point we work on the algebra of the budget constraint to simplify the expression and reduce the computational burden. We first decompose the last line

$$\ldots \frac{D_{a-1,t-1}}{D_{t-1}} P_t^D \left[D_t - \left(1-\delta_t^d\right) D_{t-1}\right] - \frac{D_{a-1,t-1}}{D_{t-1}} P_t^{Land} q_t^{LS} - \frac{D_{a-1,t-1}}{D_{t-1}} P_t^I q_t^I$$

and eliminate the terms in $\left(1-\delta_t^d\right)$ to obtain

$$\begin{aligned}
B_{a,t} &= B_{a-1,t-1} + r_{a,t} B_{a-1,t-1} + \tilde{y}_{a,t} - p_t^c c_{a,t} - (1-\mu_{a,t})\, P_t^D D_{a,t} \\
&- \left\{ \left(1 + r_t^{mort}\right) \mu_{a-1,t-1} + \tau_t^W + x_t \right\} P_{t-1}^D D_{a-1,t-1} + \alpha_t^{Land} P_{t-1}^D D_{a-1,t-1} \\
&+ P_t^D D_{a-1,t-1} \left[\frac{D_t}{D_{t-1}}\right] - \frac{D_{a-1,t-1}}{D_{t-1}} P_t^{Land} q_t^{LS} - \frac{D_{a-1,t-1}}{D_{t-1}} P_t^I q_t^I
\end{aligned}$$

now use the fact that α_t^{Land} is given by

$$\alpha_t^{Land} = \Omega_t^{Land} \frac{P_t^{Land} q_t^{LS}}{P_{t-1}^D} = \frac{1}{D_{t-1}} \frac{P_t^{Land} q_t^{LS}}{P_{t-1}^D}$$

[46] The household model is designed to capture the majority of transactions which involve already built housing. The possibility that households themselves buy land and contract the building of the house is partially captured by the intermediary and the inclusion of its profits in the budget constraint.

and replace so that the land terms disappear completely and obtain a budget constraint with fewer terms

$$B_{a,t} = B_{a-1,t-1} + r_{a,t} B_{a-1,t-1} + \tilde{y}_{a,t} - p_t^c c_{a,t}$$
$$- (1-\mu_{a,t}) P_t^D D_{a,t} - \left\{ \left(1 + r_t^{mort}\right) \mu_{a-1,t-1} + \tau_t^W + x_t \right\} P_{t-1}^D D_{a-1,t-1}$$
$$+ \left[P_t^D D_t - P_t^I q_t^I \right] \frac{P_{t-1}^D D_{a-1,t-1}}{P_{t-1}^D D_{t-1}}$$

HTM households. The budget constraint is

$$y_{a,t} = p_t^c c_{a,t} + f\left(D_{a,t}, D_{a-1,t-1}\right)$$

Optimal decision

$$D_{a,t} - \chi^D D_{a-1,t-1} = \lambda_{a,t}^D \cdot \left(C_{a,t} - \chi^C C_{a-1,t-1}\right) \cdot \left(\frac{P_t^D}{P_t^C}\right)^{-\eta}$$

For these households the budget constraint with correction for intermediary profits is

$$y_{a,t} = p_t^c c_{a,t} + (1-\mu_{a,t}) P_t^D D_{a,t}$$
$$+ \left\{ \left(1 + r_t^{mort}\right) \mu_{a-1,t-1} + \tau_t^W + x_t \right\} P_{t-1}^D D_{a-1,t-1}$$
$$+ \left[P_t^I q_t^I - P_t^D D_t \right] \frac{P_{t-1}^D D_{a-1,t-1}}{P_{t-1}^D D_{t-1}}$$

CODE. The cost of housing in the budget constraint is coded including intermediary profits. For that purpose we define the object

$$X_{a,t}^F = \left(1 + r_t^{mort}\right) \mu_{a-1,t-1} + \tau_t^W + x_t + \frac{P_t^I q_t^I - P_t^D \cdot D_t}{P_{t-1}^D \cdot D_{t-1}}$$

which can be used in both the HTM and optimizing agents budget constraints, so that the cost of housing becomes

$$F_{a,t} = (1-\mu_{a,t}) \cdot P_t^D \cdot D_{a,t} + X_{a,t}^F \cdot P_{t-1}^D \cdot D_{a-1,t-1} \cdot \Gamma_{a,t} \neq f_{a,t}$$

Note that $F_{a,t} \neq f_{a,t}$ so that the object $f_{a,t}$ which we care about in terms of economic analisys is not the same as the compact object $F_{a,t}$ used in the budget constraint.

2.11.7 Wealth in the utility function

An example is useful to understand that wealth in utility helps fit the discount factor in the household model. Consider the representative agent problem with log utility and 100% capital depreciation, and extend it with utility of capital:

$$U = log\left(AK_t^{\alpha} - K_{t+1}\right) + \gamma log\left(K_t\right)$$

If $\gamma = 0$ this problem has the well known solution $K_{t+1} = \alpha\beta AK_t^{\alpha}$ where β is the discount factor. With $\gamma > 0$ the solution becomes

$$K_{t+1} = \frac{\alpha\beta + \gamma\beta}{1 + \gamma\beta} AK_t^{\alpha}$$

and comparing terms there is more investment if $1 > \beta\alpha$, which is always true.

So, the presence of $\gamma > 0$ allows the model to generate a bigger capital stock. On the other hand, if we want to fit the same data - and assume we keep the same value of α - then we have a relationship between a new value of β, the value of beta in the standard model which we relabel β_0, and the new (non zero) value of γ, such that we obtain the same investment ratio:

$$\frac{\alpha\beta + \gamma\beta}{1 + \gamma\beta} = \beta_0\alpha$$

The new β is now a function of the "old" β_0 and of γ, subject to this restriction

$$\beta\left(\beta_0, \gamma, \alpha\right) = \frac{\beta_0\alpha}{\alpha + \gamma - \beta_0\alpha\gamma} = \beta_0 \frac{\alpha}{\alpha + \gamma\left(1 - \beta_0\alpha\right)} < \beta_0$$

The discount factor will be smaller, $\beta < \beta_0$, meaning the discount rate will be bigger. This makes sense: if we have extra utility on capital we have extra utility on the future, and if we want to have the same outcomes we must discount the future more.

This reasoning applies if we pick a constant value of γ and we adjust the new β to any values the old $\beta_0$may have. On the other hand, we can pick a constant new β and adjust γ to any values the old $\beta_0$may have. In this case the restriction is imposed by fitting γ

$$\gamma\left(\beta_0, \beta, \alpha\right) = \frac{\alpha}{\beta} \frac{\beta_0 - \beta}{1 - \beta_0\alpha}$$

subject to values $\beta < \beta_0$. In the MAKRO life cycle we use a hybrid approach where we minimize the age variability of the discount factor.

2.11.8 The multiplier effects of leverage.

The presence of the mortgage contract generates a leverage effect in the model. Specifically, when house prices rise, $p_t^D > p_{t-1}^D$, the existing debt obligation is valued at prices p_{t-1}^D but now the equity on the house is valued at the house price p_t^D. It may be profitable to liquidate the previous mortgage, sell the house and buy a bigger house using the fact that one only has to commit a small fraction of funds because one is allowed to borrow. This mechanism is better understood if we look explicitly at the cost of housing object f in the budget constraint.

$$f\left(D_{a,t}, D_{a-1,t-1}\right) = \left(1 - \mu_{a,t}\right) P_t^D D_{a,t}$$

$$+ \left\{\left(1 + r_t^{mort}\right)\mu_{a-1,t-1} + \tau_t^W + x_t - \frac{P_t^D}{P_{t-1}^D}\left(1 - \delta_t^d\right) - \alpha_t^{Land}\right\} P_{t-1}^D D_{a-1,t-1}$$

We can rearrange this into

$$f = \underbrace{(1-\mu_{a,t})\, P_t^D D_{a,t}}_{\text{Out of pocket: new house equity}} - \left[\underbrace{\left(1-\delta_t^d\right) \frac{P_t^D}{P_{t-1}^D} - \mu_{a-1,t-1}}_{\text{exisiting house equity ratio}} \right] P_{t-1}^D D_{a-1,t-1}$$

$$+ \underbrace{\left\{ r_t^{mort} \mu_{a-1,t-1} + \tau_t^W + x_t - \alpha_t^{Land} \right\} P_{t-1}^D D_{a-1,t-1}}_{\text{Unavoidable net carrying costs}}$$

The key feature is that an increase in house prices has a marginal effect which is not dragged down by the previous debt $\mu_{a-1,t-1}$. We have

$$\frac{\partial f}{\partial p_t^D} = \underbrace{(1-\mu_{a,t})\, D_{a,t}}_{\text{effect on new house equity}} - \underbrace{\left(1-\delta_t^d\right) D_{a-1,t-1}}_{\text{effect on exisiting house equity}}$$

and since $1-\delta > 1-\mu$ the cost of housing comes down when house prices increase. Therefore it is possible to buy extra housing.

Notice that there are no transaction costs which implies taking advantage of the leverage effect is costless. This potentially makes the leverage effect very powerful.

Now, this mechanism here is static. Rational agents are forward looking so they will not rush to buy more houses if prices are likely to fall in the future, which will happen if the cause of the increase in house prices is a temporary shock. As they antecipate capital losses they will dampen their current response to the price increase. That is not the case, however, for HTM agents. So the leverage effect will be active mainly in these agents.

Financial accelerator

The leverage effect is **not** the financial accelerator effect of Kyotaki and Moore, or of Bernanke and Gertler. In fact, the mortgage contract **worsens** with an increase in house prices, $\partial\mu/\partial p < 0$, which makes it a stabilizer rather than an accelerator. An increase in house prices, even though it raises the value of your current house (your collateral) does not relax the mortgage financial constraint but rather tightens it. It does, however, allow the household to exploit an available (slightly worse) contract and buy more houses simply because the household now has more money and because **there exists** an available debt contract.

Yet, MAKRO does have an accelerator in the KM and BG sense. It lies in the utility from leaving a bequest. This object is a concave function of the sum $B + p\,(1-\mu)\,D$. This combined object has an admissible lower bound. If the household is near this lower bound, an increase in house prices allows liquid wealth B to decrease, which allows households to consume more and buy more houses. The constraint has been relaxed by the house price increase, and here, buying extra housing relaxes the constraint next period also. This dynamic effect has all the hallmarks of the classic financial accelerator mechanism.

3 Firms

In addition to the public sector there are eight private sectors in the economy, indexed by the subscript *sp*. These are agriculture (including fishing), construction, energy provision, extraction, housing, manufacturing (including food processing), sea transport, and services (excluding sea transport).[47]

Firms maximize the present discounted value of profits, where the discount factor reflects a financial arbitrage condition for equity investors. Solving this problem requires both cost minimization and optimal price setting. As explained in the pricing chapter these two problems are separated into two sub-sectors - an intermediate sub-sector actually producing the goods and choosing inputs optimally, and another sub-sector where retail firms buy goods from producers, set prices, and sell the same goods to the final consumers. In the documentation (and code) the production and price setting decisions are separated. The production problem is given in this chapter and the optimal price setting problem is described in the pricing chapter.

All private sector production firms in the model use labor, capital, and materials as inputs. These inputs generate output through a production function which is a CES tree with different levels. Capital and materials can be bought from other domestic firms or imported. Labor services are bought from supplying households. The market for material inputs is a spot market, with a spot price, and the optimal decision is a static one. The optimal decisions for labor and capital are dynamic and the relevant price measures are user costs derived from intertemporal first order conditions for optimality.

The user cost of labor is derived in the labor market chapter. The user cost of capital is derived here. Given the correct user cost measures the problem of the firm can be solved by a sequence of cost minimization problems at every level of the CES tree. The two bottom levels of the CES tree determine input demand for materials and investment goods first from all producing sectors, and, at the very bottom, within each sector whether the input is imported or produced domestically. These two lower levels are separated in the code away from the problem of the firm and into the input-output system of market clearing relationships, by interpreting them as zero profit intermediate transformation sectors with constant returns to scale technologies. For that reason they are described both here and in the input-output chapter.

Finally, at several levels of the CES tree and in different sectors we have zero elasticities of substitution implied by the empirical work. At the end of this chapter we have an appendix that details how the equations used to solve the CES problem also apply to the limit case of zero elasticity.

This section delivers two of the five major demand components - namely material inputs,$R_{sp,t}$, and investments, $I_{i,s,t}$ - to the Input/-Output chapter as well as labor demand, L_t, to the labor market chapter.

The rest of this chapter is organized as follows:

- Cost minimization: contains a description of the production function, the CES tree, and the general cost minimization problem.
- Dynamic Optimization: contains a description of the dynamic optimization problem and the computation of the user cost of capital.
- Appendices: contain extra derivations, the description of equations and parameters as they are named and appear in the code, and data details as well as details on how the different parameters in the model are obtained.. The reader familiar with the model can go directly to this section.

[47]In the code the sector labels are in Danish and are 'off' for the public sector, and then respectively, 'lan', 'byg', 'ene', 'udv', 'bol', 'fre', 'soe', 'tje'.

3.1 Cost minimization

It is useful to discuss the cost minimization problems first. These are static optimization problems which take user costs and prices as given.

3.1.1 The production function

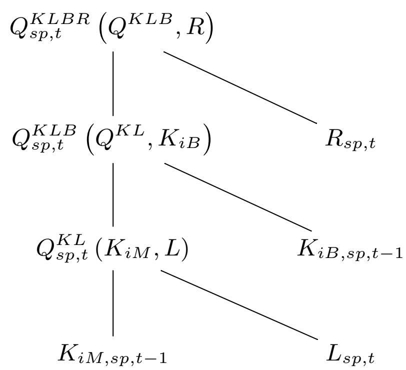
Gross output Q is produced with inputs of materials R, capital structures (buildings) K_{iB}, machinery capital goods K_{iM}, and labor, L. Capital stocks are subject to a one period time to build which implies they are fixed in the short run (current period) although they can be used with varying intensity. We write the general production function in sector sp at time t as

$$Q_{sp,t}^{KLBR} = Q\left(K_{iM,sp,t-1}, L_{sp,t}, K_{iB,sp,t-1}, R_{sp,t}\right)$$

3.1.2 The CES tree

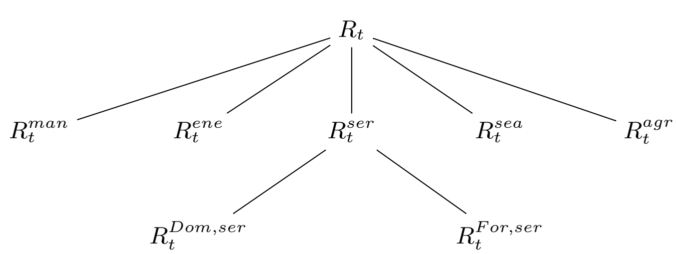
Upper level

Within the production function the different inputs come together in the following CES nest structure:



Bottom level

For materials and for capital goods there are another two levels of this tree which we detail in the input-output chapter. For these extra two levels, the upper level optimizes demand across sectors, with an identical elasticity of substitution for goods of all sectors (inputs coming from agriculture and services have the same substitutability as inputs coming from agriculture and construction). Then, the lower level optimizes demand of inputs from a given sector (say services) across domestic and foreign suppliers (if both exist). Here is a truncated illustration of the materials bottom tree:



The bottom level of the tree is slightly different for materials than for investment because materials are a flow, whereas capital is a stock. In the case of the two capital goods (buildings and machinery), these two lower levels are organized in exactly the same way as in the case of materials, but they determine the optimal composition of the *investment* flow rather than of the capital stock. The *magnitude* of the investment flow is then determined from the solution of the forward looking problem which determines the user cost and the optimal size of the stock.

Input prices

It is because of these two lower levels that the input prices that appear in the upper level of the tree are sector specific prices indexed by the demand side. In the more general formulation, all sectors (in the limit all firms) have their own slightly different demand compositions and therefore their own idiosyncratic prices. There are 8 such domestic private sector output prices and 8 foreign private sector output prices, so that input prices can be an aggregate of 16 to 18 original output prices (if we include domestic and foreign public goods).

This is also the reason why the input price of the investment good is called (and indexed) an investment price, as it is a two-layered CES aggregate of the original output prices coming out of producers in the different sectors.

The only input with a single price is labor, and yet even in this case its user cost will generally differ across sectors.

3.1.3 CES cost minimization

The optimal demand for inputs is obtained from solving a sequence of cost minimization problems at every level in the tree. As an example, the problem at the bottom of the tree is to minimize total cost $P^{KL}Q^{KL} = p^l L + p^k K$ subject to $Q^{KL} = CES\,(K, L)$. The solution to this problem is well known and yields the following objects which translate appropriately to all levels of the tree:

$$\text{Output} \quad \Rightarrow \quad Q^{KL} = Q = \left[\left(\mu^k\right)^{\frac{1}{\eta}} \left(z^k K\right)^{\frac{\eta-1}{\eta}} + \left(\mu^l\right)^{\frac{1}{\eta}} \left(z^l L\right)^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

$$\text{Derivative} \quad \Rightarrow \quad \frac{\partial Q^{KL}}{\partial L} = z^l \left(\frac{\mu^l}{z^l} \frac{Q}{L} \right)^{\frac{1}{\eta}}$$

$$\text{Demand/F.O.C.} \quad \Rightarrow \quad z^l L = \mu^l Q \left(\frac{P}{p^l} z^l \right)^{\eta}$$

$$\text{CES Price} \quad \Rightarrow \quad P^{KL} \equiv P = \left[\mu^k \left(\frac{p^k}{z^k} \right)^{1-\eta} + \mu^L \left(\frac{p^l}{z^l} \right)^{1-\eta} \right]^{\frac{1}{1-\eta}}$$

In these equations the parameters μ^j are calibrated scale parameters. The parameter η is the elasticity of substitution between the two inputs. The variable z^j is here a catch-all term that includes exogenous productivity as well as endogenous factor utilization, and in the case of labor also endogenous vacancy posting costs. The input prices p^j (not the CES prices P) are user costs except for materials where it is a CES aggregate of spot prices.

In the exact implementation of this problem at different levels in the tree some of the productivity terms z^j will be expanded while others will be eliminated (set to 1). The problem and solution remain unchanged.

One detail to mention is that, although we show it explicitly here, the production function itself is never used in the solution to the problem. Much like the utility function in the household problem, only its derivatives are ever needed, and they enter the problem through the demand functions shown above. The problem is solved using only the demand functions and the constraint in the form $PQ = p^l L + p^k K$.

Solving all the problems in the tree requires knowing the correct prices of every input. Finding the correct input prices of capital and labor involves solving a dynamic forward looking optimization problem.

The appendix provides details of all these equations as they look in the code.

3.2 Dynamic optimization

The section derives the user cost of capital. All variables and parameters in the problem generally have a sub-index (k, sp, t). In this section this index will be truncated to only t unless otherwise explicitly stated. Given eight different private sectors and two types of capital there are sixteen different versions of the variables and equations described.

3.2.1 Definitions

All capital stocks have a law of motion of the following form

$$K_t = (1 - \delta_t) K_{t-1} + I_t$$

where δ_t is the depreciation rate and I_t is the investment flow.[48]

Capital stocks are subject to Installation/Adjustment Costs given by

$$AC_t = \frac{\gamma}{2} K_{t-1} \left(\frac{I_t}{K_{t-1}} - \xi_t \frac{I_{t-1}}{K_{t-1}} \right)^2$$

where γ and ξ_t are parameters.

Capital stocks are fixed in the short run due to one period time to build, but they can be used with varying intensity u_t. The problem of optimal capital utilization is examined in the appendix. Adjustment and utilization costs as well as vacancy costs are not explicitly measured in the data, and are modeled as unobserved lost production. These costs are subtracted from gross production, Q. Gross and net output are related by:

$$Y_{sp,t} = Q^{KLBR}_{sp,t} - \sum_k AC_{k,sp,t}$$

and we do not see the vacancy posting costs or labor utilization costs as they are modeled directly inside gross output Q as detailed in the labor market chapter.

3.2.2 The discount factor

Holding an asset over one period yields the income generated by the asset and the capital gain over the period. Arbitrage implies

$$r_t V_{t-1} = Income_t + V_t - V_{t-1}$$

such that income and capital gains adjust endogenously to fit this equality. In the absence of shocks to the economy the rate of return r_t is also the required rate of return which investors demand. In the presence of shocks this arbitrage condition breaks down momentarily as the realized rate of return will differ from the required return in the moment of impact of the shock.

This arbitrage mechanism applies to all assets, and in our case to the equity of the firm. It is assumed that all private firms are owned by stockholders. The rate of return required by stockholders is taken as given by the firm. It may vary across sectors if these have different risk premia. The discount factor for the cash flows generated by the firm is defined by this rate, $\beta_t = 1/(1 + r_t)$. We work through the details of the financial side of the firm separately, where we also discuss substantial taxation and corporate capital structure issues.

[48] Inventory investment is assumed to be proportional to net output: $I_{invt,sp,t} = \mu^{Invt}_{sp,t} Y_{sp,t}$. It does not accumulate or contribute to production. It is just a drain on resources in order to match the model with national accounts data, where it is small fraction of total spending (less than 0.5 percent). Inventories are listed in the index k which identifies the three types of capital (inventories, machinery, buildings): $\{k = invt, iM, iB\}$.

The discount factor above applies to the unconstrained firm. At the end of this chapter we discuss the introduction of costly external finance in the problem of the firm, which is modeled so that its effect is contained in an augmented discount factor for the firm.

3.2.3 The problem of the firm

The problem we look at here is identical to the actual problem being solved by the firms in MAKRO. For exposition purposes we state the problem with a single capital stock rather than two, and we use only the time index on all variables. As we focus only on the optimal choice of capital we also leave out many details of the labor input. The appendix shows in more detail the problem used in MAKRO. The operating surplus π in a given period is given by

$$\pi_t = (1-\tau_t)\left(\begin{array}{c} P_t Y_t - P_t^R R_t - \left[1+\tau_t^L\left(1-\mu_t^{SEMP}\right)\right] w_t L_t \\ -\tau_t^K P_t^I K_{t-1} - T_t \end{array}\right)$$

$$\begin{gathered} -r_t^{Debt}\left(1-\tau_t\right)\mu_{t-1}^{Debt} P_{t-1}^I K_{t-1} \\ -P_t^I I_t + \tau_t \delta_t^{Tax} K_{t-1}^{Tax} + \mu_t^{Debt} P_t^I K_t - \mu_{t-1}^{Debt} P_{t-1}^I K_{t-1} \\ +q_t\left(\left(1-\delta_t\right) K_{t-1} + I_t - K_t\right) \\ +q_t^{Tax}\left(\left(1-\delta_t^{Tax}\right) K_{t-1}^{Tax} + P_t^I I_t - K_t^{Tax}\right) \end{gathered}$$

where net output $Y_t$is given by

$$Y_t = Q_t\left(u_t K_{t-1}, R_t, L_t\right) - \frac{1}{2}\gamma K_{k,t-1}\left(\frac{I_t}{K_{t-1}} - \xi_t \frac{I_{t-1}}{K_{t-1}}\right)^2$$

To locate properly these elements in the above tree, the production function Q here corresponds to the top of the tree, KLBR, and the optimization price P is also the top price which in the code is again indexed by KLBR.

The first block of the surplus expression in curved brackets lists elements affected by corporate taxation τ_t. It contains net output, minus expenses on materials and on labor costs. These last ones contain an input tax adjustment for the self employed (which we do not model separately).[49] Then we have taxes on capital goods and a lump sum production tax T_t.

First after these terms, and also affected by corporate taxation, are the costs of servicing corporate debt (the debt which is part of corporate capital structure and which is assumed to be proportional to the physical capital of the firm).

Then come the nominal investment cost, the value of the tax deduction from capital depreciation, revenues (expenses) from increases (reductions) in corporate debt, and finally the Lagrange multiplier (Tobin's q) attached to the law of motion for capital in real terms (the standard one), and in tax (or book) value. The tax value of capital is the nominal object, K^{Tax}.

[49] τ^K and τ^L are input taxes. The labor tax falls on *hired* labor (i.e. excluding the self employed).

First order conditions

The discount factor between time t and time $t+1$ is given by $\beta_{t+1} = \frac{1}{1+r_{t+1}}$. The optimal choice of labor is dynamic and detailed in the labor market chapter. The first order condition for capital utilization is discussed in the appendix. The first order condition for materials, R_t, is given by $P_t \frac{\partial Q_t}{\partial R_t} = P_t^R$ and in fact this equation is never used as it is redundant given that it is identical to what is obtained in the CES cost minimization problem above. The first order condition for investment, I_t, which isolates Tobin's q, is

$$q_t = P_t^I \left(1 - q_t^{Tax}\right) + \Gamma_t - \frac{\xi_{t+1}}{1+r_{t+1}} \Gamma_{t+1}$$

$$\Gamma_t \equiv P_t \left(1 - \tau_t\right) \gamma \left(\frac{I_t}{K_{t-1}} - \xi_t \frac{I_{t-1}}{K_{t-1}} \right)$$

where we see that an increase in current investment raises costs today but also allows for greater investment at a lower cost tomorrow.

The first order condition for the book/tax value of capital, K_t^{Tax} , is

$$q_t^{Tax} = \frac{\tau_{t+1} \delta_{t+1}^{Tax}}{\left(1 + r_{t+1}\right)} + \frac{\left(1 - \delta_{t+1}^{Tax}\right)}{\left(1 + r_{t+1}\right)} q_{t+1}^{Tax}$$

where δ^{Tax} is the tax deductible depreciation rate. The tax deduction comes only after one period, due to the time investment takes to depreciate.[50] We can see that this Lagrange multiplier is given by a Bellman equation which computes the present discounted value of all future tax benefit revenues.

The first order condition for capital, K_t, is

$$P_{t+1} \frac{\partial Y_t}{\partial K_t} - \tau_{t+1}^K P_{t+1}^I =$$

$$q_t \frac{\left(1 + r_{t+1}\right)}{\left(1 - \tau_{t+1}\right)} - \frac{\left(1 - \delta_{t+1}\right)}{\left(1 - \tau_{t+1}\right)} q_{t+1} - \mu_t^D P_t^I \frac{\left(r_{t+1} - r_{t+1}^D \left(1 - \tau_{t+1}\right)\right)}{\left(1 - \tau_{t+1}\right)}$$

and from the derivative of net output we isolate the user cost of capital

$$P_{t+1} \frac{\partial Y_{t+1}}{\partial K_t} \equiv \underbrace{P_{t+1} \frac{\partial Q_{t+1}}{\partial \left(u_{t+1} K_t\right)} u_{t+1}}_{P_{t+1}^K\text{:user cost of } K_t} - P_{t+1} \frac{\partial AC_{t+1}}{\partial K_t}$$

Some intuition is immediate. The corporate tax rate raises the user cost of capital by the factor $1/(1-\tau)$ through Tobin's q. Having corporate debt reduces the user cost of capital as the cost of this debt r_t^{Debt} is lower than the cost of equity funding r_{t+1}. And not surprisingly, taxes on capital τ^K raise the user cost. The last term measures how an increase in K_t, decided in period t, lowers installation costs in period $t+1$.

3.3 The firm as a financial entity

Firms do not just trade in physical capital, labor services, and intermediate inputs, in order to produce and sell output. They also hold assets which are not directly related to their production activity. Firms hold such assets due to the existence of financial frictions. These frictions are currently modeled minimally in the model and so the presence of

[50] This may not be fully consistent with accounting practices.

financial assets in the balance sheet of the firm is dealt with in reduced form and not as part of an explicit optimization decision.

Another reason we observe assets inside firms in our model is the scope of aggregation in the data. Aggregation is both vertical (from firms into sectors) and horizontal (which types of firms are included within each sector). The latter bundles together production firms with financial firms such as investment funds, and for this reason also, the production and financial parts of our firms are separated. Nevertheless, these assets must be properly accounted for so that we fit the national accounts data (Nationalregnskabet) on the side of firms just as accurately as we do for households and for the government.

In order to do so we need to define two main objects. One is the discount rate r_t applied to the income flow y_t generated by the firm, and the other is the income variable itself. Financial assets will then enter the income variable exogenously, and make up a separate portion of firm value from that created by endogenous production decisions.

3.3.1 The discount rate

The discount rate is the rate required by investors in order to own equity in the firm. The income generated by the firm is discounted by this rate. Standard arbitrage then links the value of the firm V and the income flow

$$V_{t-1} = \frac{y_t + V_t}{1 + r_t}$$

and the expected return on equity equals a normal return on bonds plus a risk premium, $r_t = r_t^{Bonds} + r_t^{rp}$.

As the firm is decomposed into independent financial and production components we can write the above value as the sum of

$$V_{t-1} = V_{t-1}^{Exo} + V_{t-1}^{Endo} = V_{t-1}^{Exo} + \frac{y_t^{Endo} + V_t^{Endo}}{1 + r_t}$$

where the superscript *Exo* denotes the financial part which is exogenous to the optimization problem of our firms, and where the superscript *Endo* denotes the endogenous operational surplus. We return to this decomposition below.

3.3.2 The income flow, financial assets, and debt

Some of the income flow comes not from production value added but from holding financial objects which we divide broadly into debt D and financial assets A.

Since there are no financial frictions a liquid financial asset A inside the firm must be valued at its current market price as it can be freely traded. Not only that, holding these assets either in positive or negative positions is equivalent to issuing new shares or buying back existing ones as the firm can borrow from and pay to shareholders without cost. Implicitly then, all assets in A earn the required return on equity. All stocks held by our firms satisfy this.

Our firms also hold cash and other low return instruments. For these assets, their value outside the firm would be higher than the discounted nominal value of their returns inside the firm. However, accounting for them this way ignores the value of their convenience yield, which is the reason they are held in the first place.[51] Non financial firms hold significant amounts of cash, with Microsoft being the company with the largest cash

[51] The convenience yield is the exact difference between the nominal yield the asset generates and the required rate on equity. Keynes used the idea of convenience yield in his money demand function. Also Del Negro et al. (2017), Safety, Liquidity, and the Natural rate of Interest, Brookings Papers, Spring 2017.

reserves in the world. Investment funds hold bonds and low return instruments for portfolio risk management. Correcting for the convenience yield allows all assets inside our firms to be valued at their current market value and decoupled from the firm's operational side.

We model debt issued by the firm differently from a negative asset. One significant component of firm debt is mortgage debt, and this is closely related to (or collateralized by) the capital stock of the firm. Corporate debt is also issued with a variety of covenants (such as not allowing sales of installed capital) which serve as an indirect claim on the firm's buildings and machinery. For these reasons we model debt as proportional to the capital stock of the firm:

$$D_t = \mu_t^D p_t^I K_t$$

where μ_t^D is a debt factor which is exogenous to the firm. This way of modeling firm debt mirrors the way mortgage debt is modeled on our household side of the model. The value of $\mu_t^D$is the expression of the modified Modigliani-Miller theorem. There is an implicit trade-off between bankruptcy risk and associated costs on one hand, and the gain from financing the firm at a lower rate on the other. Currently all firms in the model have the same constant debt factor $\mu_t^D = 0.4$. As this debt is a function of the capital stock of the firm, it is endogenous. Assets A_t are exogenous although accounting properly for them requires detail.

3.3.3 The income flow: revenues and expenses

We start by listing the revenues earned and costs incurred by firms. Selling (net) output, holding financial assets, and borrowing from outside the firm, all increase the amount of cash inside the firm. We divide assets into two types, $A = A^S + A^B$ in order to discriminate their nominal returns and tax treatment. Income generated by these sources is given by:

$$P_t^Y Y_t + r_t^S A_{t-1}^S + r_t^B A_{t-1}^B + A_{t-1} - A_t + D_t - \left(1 + r_t^D\right) D_{t-1}$$

One last source of income is the capital depreciation exemption from corporate taxation, where capital is valued with a tax reference method. This closes the revenue side and is given by

$$\tau_t \delta_t^{Tax} K_{t-1}^{Tax}$$

Now, the following objects drain resources from the firm: wage payments, investment costs, intermediate input costs, and input specific taxes (τ_t^L, τ_t^K) as well as other non corporate taxes or transfers T. Together these are

$$\hat{w}_t L_t + P_t^I I_t + P_t^R R_t + \tau_t^K P_t^I K_{t-1} + T_t$$

where $\hat{w}_t = w_t + w_t \tau_t^L (1 - r_t^{self})$ includes employment taxes on hired (not self employed) labor, where r_t^{self} is the fraction of self employed. We have then a preliminary expression for income before corporate taxation:

$$\hat{y}_t = P_t^Y Y_t + D_t - \left(1 + r_t^D\right) D_{t-1} + \tau_t \delta_t^{Tax} K_{t-1}^{Tax}$$

$$+ r_t^S A_{t-1}^S + r_t^B A_{t-1}^B + A_{t-1} - A_t$$

$$- \hat{w}_t L_t - P_t^I I_t - P_t^R R_t - \tau_t^K P_t^I K_{t-1} - T_t$$

where assets A are exogenous to the optimization problem of the firm. The last step to write the income which is relevant for shareholders is to define the scope of corporate taxation τ_t.

3.3.4 EBITDA

In order to define the scope of corporate taxation τ_t we make a detour to discuss how the concepts of Earnings Before Interest, Tax, Depreciation and Amortization (EBITDA), and Earnings Before Tax (EBT), translate into the income generated by our firm.[52] We start with

$$EBITDA_t = P_t^Y Y_t - w_t L_t - P_t^R R_t - T_t^{NP}$$

which is production minus wages and intermediate input costs, and minus net production taxes. Net production taxes are a collection of different objects, here land taxes, weight taxes on vehicles, payroll taxes and others. Some of these objects, for which we have total sums in the data, are then modeled as functions of the firm's variables:

$$T_t^{NP} = \underbrace{\tau_{b,t}^k P_{b,t}^I K_{b,t-1}}_{T_t^{Land}} + \underbrace{\tau_{m,t}^k P_{m,t}^I K_{m,t-1}}_{T_t^{Weight}} + \underbrace{\tau_t^L w_t \left(1 - r_t^{Self}\right) L_t}_{T_t^{Payroll}} + T_t^{Rest}$$

where land taxes are written in terms of the buildings stock of the firm, the vehicle weight tax is written as a function of the stock of machinery, and the payroll tax is written as a function of employment.

We have been writing the problem with a single capital stock, and we will continue to do so now, and therefore the capital tax is understood to be the applicable one when we think of capital as buildings or machinery. We now write EBITDA again:

$$EBITDA_t = P_t^Y Y_t \underbrace{-w_t L_t - \tau_t^L w_t \left(1 - r_t^{Self}\right) L_t}_{-\hat{w}_t L_t} - P_t^R R_t - \tau_t^K P_t^I K_{t-1} - T_t^{Rest}$$

We are missing interest and depreciation in the earnings expression.

3.3.5 EBT

Adding interest and depreciation to the problem yields earnings before taxes:

$$EBT_t = EBITDA_t - \delta_t^{Tax} K_{t-1}^{Tax} + r_t^B A_{t-1}^B - r_t^D D_{t-1}$$

where only assets of type B are subject to corporate taxes on their nominal income.

3.3.6 Corporate taxes

We are at the last step now. The corporate tax term falls on EBT:

$$T_t^{Virk} = \tau_t EBT_t$$

Given EBT and EBITDA, the income flow relevant for shareholders is:

$$y_t = EBITDA_t - T_t^{Virk} - P_t^I I_t + T_t^0$$

[52] These concepts are necessary because in some periods there is available data for them while some detailed data is lacking, thus they allow for a complete description of the balance sheet of the firm. The tax implicitly left out of EBITDA and EBT is the corporate tax.

$$+r_t^S A_{t-1}^S + r_t^B A_{t-1}^B + A_{t-1} - A_t + D_t - \left(1 + r_t^D\right) D_{t-1}$$

where T^0 includes a number of transfers and other operations between firms and government and firms and households.

3.3.7 The income flow and dynamic optimality

Inserting terms from EBITDA, EBT, and corporate taxes we obtain:

$$y_t = (1 - \tau_t) \left[P_t^Y Y_t - \hat{w}_t L_t - P_t^R R_t - \tau_t^K P_t^I K_{t-1} - T_t^{Rest}\right]$$
$$+\tau_t \left[r_t^D D_{t-1} + \delta_t^{Tax} K_{t-1}^{Tax} - r_t^B A_{t-1}^B\right] - P_t^I I_t$$
$$+r_t^S A_{t-1}^S + r_t^B A_{t-1}^B + A_{t-1} - A_t + D_t - \left(1 + r_t^D\right) D_{t-1}$$

$$+T_t^0 + q_t \left[(1 - \delta_t) K_{t-1} + I_t - K_t\right] + q_t^{Tax} \left[\left(1 - \delta_t^{tax}\right) K_{t-1}^{tax} + P_t^I I_t - K_t^{tax}\right]$$

where in the bottom row we set the exogenous lump sum transfer term apart, and add the Lagrange multipliers on the laws of motion for capital and tax capital. Now use the modeling assumption that debt is proportional to capital, $D_t = \mu_t^D P_t^I K_t$, and take derivatives with the discount factor $\beta_{t+1} = \frac{1}{1+r_{t+1}}$ to obtain the optimality condition for capital:

$$P_{t+1}^Y \frac{\partial Y_{t+1}}{\partial K_t} = \tau_{t+1}^K P_{t+1}^I +$$
$$+q_t \frac{(1 + r_{t+1})}{(1 - \tau_{t+1})} - \frac{(1 - \delta_{t+1})}{(1 - \tau_{t+1})} q_{t+1} - \mu_t^D P_t^I \frac{\left(r_{t+1} - r_{t+1}^D (1 - \tau_{t+1})\right)}{(1 - \tau_{t+1})}$$

and the last term on the right hand side shows that the bigger the gain from debt financing, the lower the desired marginal product of capital and the bigger the capital stock.

3.3.8 Assets and the value of the firm

When looking at data we consider *broadly* two types of assets, stocks A^S and bonds, A^B where stocks earn exactly the equity rate, $r^S = r$, and are tax free, and bonds earn a lower rate $r^B < r$ and are subject to taxation.[53] We have then

$$\underbrace{-\tau_t \left[r_t^B A_{t-1}^B\right] + \left(1 + r_t^B\right) A_{t-1}^B - A_t^B}_{y_t^{AB}} + \underbrace{(1 + r_t) A_{t-1}^S - A_t^S}_{y_t^{AS}}$$

Stocks A^S are trivially discounted down to current face/market value. As the transversality condition sets the discounted limit to zero we have

$$V_{t-1}^{AS} = \frac{y_t^{AS}}{1 + r_t} + \frac{1}{1 + r_t} \frac{y_{t+1}^{AS}}{1 + r_{t+1}} + ... = A_{t-1}^S$$

while bonds A^B are held only if their convenience yield r_t^{CYB} obeys

$$r_t^B (1 - \tau_t) + r_t^{CYB} = r_t$$

which must be the case in equilibrium. Correcting for this A^B also discounts to its face/market value, $V_{t-1}^{AB} = A_{t-1}^B$. Alternatively, as there are no frictions preventing trade

[53] Income from stocks is tax free because taxes on dividends are paid by the issuing firm.

this is the only possible valuation for such assets. The absence of frictions also implies that cash, gold, bank deposits, bonds and stocks held as assets, dividends, and share issues or buybacks, are all perfect substitutes and all fall under the umbrella of A_t. And A_t just scales up the value of the firm one to one.[54] We can therefore define the value of the firm as the value of its endogenous operating surplus plus the face value of its exogenous assets.

$$V_{t-1} = \frac{y_t + V_t}{1 + r_t} = V_{t-1}^{Exo} + V_{t-1}^{Endo} = \underbrace{A_{t-1}^S + A_{t-1}^B}_{A_{t-1}} + V_{t-1}^{Endo}$$

where

$$V_{t-1}^{Endo} = \frac{y_t^{Endo} + V_t^{Endo}}{1 + r_t}$$

and y_t^{Endo} is the endogenous operational surplus flow[55]

$$y_t^{Endo} = (1 - \tau_t) \left[P_t^Y Y_t - \hat{w}_t L_t - P_t^R R_t - \tau_t^K P_t^I K_{t-1} - T_t^{Rest} \right]$$

$$+ D_t - \left(1 + r_t^D\right) D_{t-1} + \tau_t \left[r_t^D D_{t-1} + \delta_t^{Tax} K_{t-1}^{Tax} \right] - P_t^I I_t + T_t^0$$

[54] In reality there are issues of control so that the equivalence is broken. In the famous leveraged buyout of Manchester United F.C. the controlling part has less than 100% of shares.

[55] There are two lump sum tax objects in this expression. In the code we use the EBITDA object by sector so we include $T_{sp,t}^{Rest}$ within our endogenous operational surplus, but the transfer object $T^0$is aggregated across all sectors and only enters expressions for aggregate surplus in the economy.

3.4 The extraction sector

The extractions sector consists of oil and gas as well as a small amount of gravel extraction. Output and prices from this sector are exogenous and based on forecasts from the Danish Energy Agency (Energistyrelsen). Given the exogenous output, production factors are endogenously chosen to minimize costs and modeled the same way as other sectors apart from the existence of an additional tax:

$$\tau_t = \tau_t^{selvskab} + \frac{y_t - y_t^{gravel}}{y_t} \tau_t^{oil}$$

All demand for domestic extraction output is scaled proportionally to match the supply, and all demand for extraction imports are scaled inversely such that the total demand for extraction does not depend on the exogenized sector output. I.e. import and domestic production of extraction are perfect substitutes and we simply allocate a share of domestic output to different types of demand and use imports to satisfy residual demand.

3.5 The housing sector

In the national accounts the housing sector produces housing *services* where the key input is a measure of the stock of houses. The sector includes both rental and privately owned housing and produces a homogeneous good priced at the rental value of housing. The rent value of owner housing is imputed based on rents of comparable rental housing.

In the model we have an endogenous decision by households on owned housing while rental housing enters the household problem exogenously. As we do not model the rental market both consumption and production of rental housing are exogenous. The stock of housing available for rent is exogenous and assumed to depend on government supported rental building projects. As the Danish rental market is highly regulated the rental price in the model is also exogenous.

Given that the rental part of housing is exogenous we need to decouple it from owned housing. In order to achieve this separation we structure production in this housing services sector using a Leontief function. The main input is the existing stock of housing ($\approx$ 75%, and there is no input of machinery capital. Note that this measure of the stock of housing excludes land, and therefore it is not the housing measure D described in the household chapter. In the data this sector accounts for services such has housing maintenance ($\approx$ 15%, which involves labor and intermediate inputs, and other services ($\approx$ 10%, mainly financial services), and for that reason it is organized as a production sector within the input-output structure of the economy.

These three inputs, capital (buildings), labor, and materials, generate output through a Leontief production function with the consequence that output, employment, and materials are then proportional to the capital stock measure. Unlike production in the other private sectors, there are no adjustment costs to capital and therefore net and gross production are the same.

Since everything is proportional to capital we can separate all inputs and output in proportion to the exogenous fraction of buildings that are owned housing (q_t^k described as "bricks" in the household chapter) and the fraction which are rental housing. This is equivalent to assuming separate but identical Leontief production technologies for privately owned housing and for rental housing, so they have identical amounts of materials and labor in proportion to their respective building capital stock.

Once the rental part of the data is separated in this way, we calculate how it behaves in existing data and forecast its use of resources in the future, which enter exogenously in the model.

3.5.1 Formalizing the problem

Net output Y_t uses buildings (K), labor (L), and materials (R):

$$Y_t = Q_t\left(K_{t-1}, R_t, L_t\right) = min\left(\frac{\phi_k}{\phi_R}R_t, min\left(\phi_k K_{t-1}, \frac{\phi_k}{\phi_l}L_t\right)\right)$$

and the Leontief function implies $Q_t = \phi_k K_{t-1}$, $L_t = \phi_l K_{t-1}$, and $R_t = \phi_R K_{t-1}$.

The capital stock K_t (here the total stock of buildings or "bricks") and the associated investment are exogenous in the data years, and in the forecast the owned housing part is endogenous while the rental part is an exogenous projection.[56] Output, employment, and the use of intermediate inputs are also exogenous in the rental part, while endogenous in the owner part, but all of them are proportional to the stock due to the Leontief structure. The definition of profits and the first order conditions to the problem then determine different auxiliary prices such as Tobin's q_t and q_t^{Tax}. As we show below, the normal first order conditions and user cost expressions apply to the Leontief case.

[56]The owned housing part of this stock is described in the household chapter in the section covering the housing intermediary.

3.5.2 Link to the household problem

As we decouple the rental housing from the owned housing we introduce them differently in the household problem. The rental housing coming out of this sector is added exogenously to the budget constraint of the household while owned housing is an optimal decision. Owned housing is bought from an intermediary that takes a flow product from the construction sector, puts it on a plot of land and sells the final combined good (bricks and land) to the household. Rental housing contains no land.[57] If we then remove the value added of land from the house bought by households we have a stock which is equivalent to the rental housing good, so that we can add them and the combined stock matches the data from the housing *services* sector described here.

In the problem of the household we can find in the budget constraint a term which accounts for expenses with housing maintenance, $x_t P^D_{t-1} D_{a-1,t-1}$. The factor x_t in this term is taken as given by the household and is the fraction of the labor and materials costs $\left[P^R_t \phi_R + \hat{w}_t \phi_l\right]$ we have in the problem above which is allocated to owned housing.

Finally, the household is the owner of the stock of owned housing and pays directly for its maintenance costs. Therefore only the fraction of sector profits that corresponds to rental housing can be allocated through the ownership of the firm in the household portfolio, controlling for the fact that households consume an exogenous amount of rental housing and pay rent on it. This is further detailed below.

[57] Land is introduced in MAKRO in this specific way in order to improve the modeling of house prices. Our user cost of owned housing specifically includes the effect of the price of land. Currently land is not included in any other product or sector.

3.6 Appendices - firms

3.6.1 Cost minimization

Here we provide the details of this part of the problem as it looks in the code. Now we can look explicitly at the demand functions in all levels of the tree.

Lower branch

At the bottom of the tree firms choose between machinery capital, $K_{'iM',sp,t}$, and labor, $q^L_{sp,t}$. In the text we use the first order condition for labor as the example so we start with it:

$$z^l L = \mu^l Q \left(\frac{P}{p^l} z^l\right)^\eta$$

and in the code it reads

$$\underbrace{\left(1 - r_t^{OpslagOmk}\right) f^{Prod}_{sp,t} r^{LUdn}_{sp,t}}_{z^L_{sp,t}} \underbrace{q^L_{sp,t}}_{L} = \underbrace{f^{KL}_{sp,t} \underbrace{\mu^L_{sp,t}}_{\text{Share}}}_{\text{Scale}} Q^{KL}_{sp,t} \left(\frac{P^{KL}_{sp,t}}{P^L_{sp,t}} z^L_{sp,t}\right)^{e^{KL}_{sp}}$$

There are a few details relative to the version used in the text. On the right hand side the scale parameter has two components: a share parameter $\mu^L_{sp,t}$ and a term $f^{KL}_{sp,t}$ which adjusts the overall factor productivity in the nest. On the left hand side the term z^L_t contains labor augmenting productivity $f^{Prod}_{sp,t}$, a labor utilization or "effort" variable $r^{LUdn}_{sp,t}$ which we discuss below, and, finally, we factor employment used in production by the fraction of labor lost (to production) while managing the hiring process, $r_t^{OpslagOmk}$. And of course $P^L_{sp,t}$ is the user cost of labor where labor is measured per hour of efficiency unit.

For machinery we have

$$\frac{1}{f^q} q^K_{'iM',sp,t-1} \underbrace{r^{KUdn}_{'iM',sp,t}}_{z^K_{'iM',sp,t}} = \underbrace{f^{KL}_{sp,t} \mu^K_{'iM',sp,t}}_{\text{Scale}} Q^{KL}_{sp,t} \left(r^{KUdn}_{'iM',sp,t} \frac{P^{KL}_{sp,t}}{P^K_{'iM',sp,t}}\right)^{e^{KL}_{sp}}$$

where $P^K_{'iM'sp,t}$ is the user cost of machinery capital, $r^{KUdn}_{'iM',sp,t}$ is the utilization of rate of capital which we discuss below, and the factor f^q is the growth correction factor for all lagged quantities in the model.

The elasticity of substitution in this branch, e^{KL}_{sp}, varies across sectors. Elasticity estimates are taken from Kronborg et al. (2020) although we set a lower bound at 0.1 in MAKRO. Sectors with elasticities higher than 0.1 are manufacturing, (0.51), services, (0.42),and extraction, (0.33).

Middle branch

One level up in the tree firms choose between buildings (structures), $K_{'iB',sp,t}$, and the aggregate of machinery and labor, $Q^{KL}_{sp,t}$, and the demand equations in the code are:

$$\frac{1}{f^q} q^K_{'iB',sp,t-1} \underbrace{r^{KUdn}_{'iB',sp,t}}_{z^K_{'iB',sp,t}} = \underbrace{f^{KLB}_{sp,t} \mu^K_{'iB',sp,t}}_{\text{Scale}} Q^{KLB}_{sp,t} \left(r^{KUdn}_{'iB',sp,t} \frac{P^{KLB}_{sp,t}}{P^K_{'iB',sp,t}}\right)^{e^{KLB}_{sp}}$$

$$Q^{KL}_{sp,t} = \underbrace{f^{KLB}_{sp,t}\mu^{KL}_{sp,t}}_{\text{Scale}} Q^{KLB}_{sp,t}\left(\frac{P^{KLB}_{sp,t}}{P^{KL}_{sp,t}}\right)^{e^{KLB}_{sp}}$$

where $P^{K}_{'iB',sp,t}$ is the user cost of capital for buildings and $P^{KL}_{sp,t}$ is the CES price index for the labor and machinery capital object. In ADAM the elasticity of substitution between buildings and other inputs is set to zero. Kronborg et al. (2020) also point to very low values of this elasticity which we set to $e^{KLB}_{sp} = 0.1$, with the salient exception of the extraction sector where $e^{KLB}_{sp} = 1.57$.

Top branch

Here firms choose between materials, $q^{R}_{sp,t}$, and the aggregate $Q^{KLB}_{sp,t}$:

$$q^{R}_{sp,t} = \underbrace{f^{KLBR}_{sp,t}\mu^{R}_{sp,t}}_{\text{Scale}} Q^{KLBR}_{sp,t}\left(\frac{P^{KLBR}_{sp,t}}{P^{R}_{sp,t}}\right)^{e^{KLBR}_{sp}}$$

$$Q^{KLB}_{sp,t} = \underbrace{f^{KLBR}_{sp,t}\mu^{KLB}_{sp,t}}_{\text{Scale}} Q^{KLBR}_{sp,t}\left(\frac{P^{KLBR}_{sp,t}}{P^{KLB}_{sp,t}}\right)^{e^{KLBR}_{sp}}$$

where $P^{R}_{sp,t}$ and $P^{KLB}_{sp,t}$ are the sector specific CES price indices for materials and for the the KLB aggregate, and $P^{KLBR}_{sp,t}$ is the global optimization price. Following estimates from Kronborg, A. (2020) the elasticity of substitution, e^{KLBR}_{sp}, is set to 0.1 for all sectors with the exceptions of manufacturing (0.53) and construction, (0.41).

Total cost identities

The cost minimization problems are solved using the demand functions and the respective total cost identities:

$$P^{KL}_{sp,t}Q^{KL}_{sp,t} = P^{L}_{sp,t}q^{L}_{sp,t} + P^{K}_{'iM',sp,t-1}q^{K}_{'iM',sp,t-1}$$

$$P^{KLB}_{sp,t}Q^{KLB}_{sp,t} = P^{K}_{'iB',sp,t-1}q^{K}_{'iB',sp,t-1} + P^{KL}_{sp,t}Q^{KL}_{sp,t}$$

$$P^{KLBR}_{sp,t}Q^{KLBR}_{sp,t} = P^{R}_{sp,t}q^{R}_{sp,t} + P^{KLB}_{sp,t}Q^{KLB}_{sp,t}$$

The upper price $P \equiv P^{KLBR}_{sp,t}$ has a special interpretation: it is the marginal cost of producing one more unit of output. The price index for materials, $P^{R}_{sp,t}$, is given in the Input/Output chapter, the user cost of labor, $P^{L}_{sp,t}$, is given in the labor market chapter.

3.6.2 Dynamic optimization

Adjustment/installation costs

In the text we have

$$\frac{\gamma}{2}K_{t-1}\left(\frac{I_t}{K_{t-1}} - \xi_t\frac{I_{t-1}}{K_{t-1}}\right)^2$$

and in code terminology we have

$$\frac{\mu_{k,sp}^{KInstOmk}}{2} q_{k,sp,t-1}^{K} \left(\frac{q_{k,sp,t}^{I} - f_{k,sp,t}^{KInstOmk} q_{k,sp,t-1}^{I} \frac{1}{f^q}}{q_{k,sp,t-1}^{K}} \times f^q \right)^2$$

In the calibration $\xi_t \equiv f_{k,sp,t}^{KInstOmk} = f^q \times q_{k,sp,t}^{I} / q_{k,sp,t-1}^{I}$ so that adjustment costs are zero in historical data, and in the projection $\xi = 1 + g$, where g_t is the Harrod neutral steady state growth rate. The adjustment cost level parameter $\gamma \equiv \mu_{k,sp}^{KInstOmk}$ is not time dependent and is estimated to match dynamic moments of investment in the data.

Net output

In the text

$$Y_t = Q_t \left(u_t K_{t-1}, R_t, L_t \right) - \frac{1}{2} \gamma K_{k,t-1} \left(\frac{I_t}{K_{t-1}} - \xi_t \frac{I_{t-1}}{K_{t-1}} \right)^2$$

and in the code

$$q_{sp,t}^{Y} = q_{sp,t}^{KLBR} - q_{'ktot',sp,t}^{KInstOmk}$$

We use here the capital letter Q for output quantities and the lower case q for input quantities to help visually, but in the code all quantities have the lower case prefix q.

F.O.C. Tax value of capital

We have in the text

$$q_t^{Tax} = \frac{q_{t+1}^{Tax} \left(1 - \delta_{t+1}^{Tax} \right) + \tau_{t+1} \delta_{t+1}^{Tax}}{\left(1 + r_{t+1} \right)}$$

In the code

$$Er_{k,sp,t}^{SkatAfskr} = \frac{Er_{k,sp,t+1}^{SkatAfskr} \left(1 - r_{k,sp,t+1}^{SkatAfskr} \right) + \tau_{t+1} r_{k,sp,t+1}^{SkatAfskr}}{1 + r_{t+1}^{VirkDisk}}$$

F.O.C. Investment. Tobin's q.

We have in the text

$$q_t = P_t^I \left(1 - q_t^{Tax} \right) + P_t^Y \left(1 - \tau_t \right) \gamma_t \left(\frac{I_t}{K_{t-1}} - \xi_t \frac{I_{t-1}}{K_{t-1}} \right)$$

$$- \frac{\xi_{t+1}}{1 + r_{t+1}} P_{t+1}^Y \left(1 - \tau_{t+1} \right) \gamma_{t+1} \left(\frac{I_{t+1}}{K_t} - \xi_t \frac{I_t}{K_t} \right)$$

and this equation in the code looks as follows

$$P_{k,sp,t}^{TobinsQ} = P_{k,sp,t}^{I} \left(1 - Er_{k,sp,t}^{SkatAfskr} \right)$$

$$+ P_{sp,t}^{KLBR} \left(1 - t_t^{Selskab} \right) \mu_{k,sp}^{KInstOmk} \left(\frac{q_{k,sp,t}^{I} - f_{k,sp,t}^{KInstOmk} q_{k,sp,t-1}^{I} \frac{1}{f^q}}{q_{k,sp,t-1}^{K}} \times f^q \right)$$

$$- \frac{f_{k,sp,t+1}^{KInstOmk}}{1 + r_{t+1}^{VirkDisk}} P_{sp,t+1}^{KLBR} \left(1 - t_{t+1}^{Selskab} \right) \times$$

$$\times \mu^{KInstOmk}_{k,sp} \left(\frac{f^q \times q^I_{k,sp,t+1} - f^{KInstOmk}_{k,sp,t+1} q^I_{k,sp,t}}{q^K_{k,sp,t}} \right)$$

Note that Tobin's q is a Lagrange multiplier on the real quantity K and therefore it is a price, so that it is given the letter P in $P^{TobinsQ}$. The object q^{Tax}_t is also a Lagrange multiplier but on the nominal quantity $K^{Tax}$and is therefore is not a price in the same sense as Tobin's q, and as such it is given the curiously different denomination $Er^{SkatAfskr}_{k,sp,t}$.

User cost of capital

The user cost of capital is given in the first order condition for capital. In the text we have

$$q_t \frac{1 + r_{t+1}}{(1 - \tau_{t+1})} - q_{t+1} \frac{(1 - \delta_{t+1})}{(1 - \tau_{t+1})} + \tau^K_{t+1} P^I_{t+1}$$

$$- \frac{\left(r_{t+1} - r^{Debt}_{t+1} (1 - \tau_{t+1})\right)}{(1 - \tau_{t+1})} \mu^{Debt}_t P^I_t = P^K_{t+1} - P^Y_{t+1} \frac{\partial AC_{t+1}}{\partial K_t}$$

which in the code is

$$P^{TobinsQ}_{k,sp,t} \frac{1 + r^{VirkDisk}_{t+1}}{1 - t^{Selskab}_{t+1}} - f^p P^{TobinsQ}_{k,sp,t+1} \frac{\left(1 - r^{Afskr}_{k,sp,t+1}\right)}{1 - t^{Selskab}_{t+1}} +$$

$$+ t^K_{k,sp,t+1} P^I_{k,sp,t+1} - \left(\frac{r^{VirkDisk}_{t+1} - \left(1 - t^{Selskab}_{t+1}\right) r^{Rente}_{'Obl',t+1}}{1 - t^{Selskab}_{t+1}} \right) r^{Laan2K}_t P^I_{k,sp,t} =$$

$$= f^p \underbrace{P^K_{k,sp,t+1}}_{\text{user cost of k}} + f^p P^{KLBR}_{sp,t+1} \frac{\mu^{KInstOmk}_{k,sp}}{2} \left(\frac{f^q q^I_{k,sp,t+1} - f^{KInstOmk}_{k,sp,t+1} q^I_{k,sp,t}}{q^K_{k,sp,t}} \right)^2$$

As we can see a number of objects are indexed (k, sp, t), but not all. One of the parameters which is not is the corporate tax rate $\tau_t \equiv t^{Selskab}_t$ which is an economy-wide object, and another is the discount rate for the firm which reflects preferences of equity investors.

The capital structure debt parameter $\mu^{Debt}_t = r^{Laan2K}_t$ is also not capital-type or sector specific. The debt share of the firm is given by $\mu^{Debt}_t = \alpha^{Mortgages2K}_t + \lambda^{FirmDebt}_t$ where $\lambda^{FirmDebt}_t$ is set to 0.4. More details on this in the chapter on firm finance.

3.6.3 Factor utilization

Intuition. Factor utilization is added to the model to help generate procyclical value added per worker. In order to counter diminishing returns across all factors of production due to capital rigidity it is necessary to compensate with a mechanism that increases total factor productivity. The equations used are flexible adaptations of the following idea. Let the firm have gross output given by a function of the type $Q_t = Q\left(u_t X_t\right)$ with a generic first order condition for optimal choice of X_t given by

$$P_t \frac{\partial Q_t}{\partial\left(u_t X_t\right)} u_t = P_t^X$$

which defines the user cost of X. Utilization is then associated with an auxiliary stock variable x which obeys the following law of motion $x_t = \left(u_t - 1\right) X_t + \lambda_t x_{t-1}$ where $\lambda_t = \lambda_0 \bar{u}_{t-1}^{\theta} / \beta_t$, and where $\bar{u}_t$ is an externality term which in equilibrium equals u_t. This law of motion has a steady state solution at $x = 0$ and $u = 1$. We replace the choice of u_t with the choice of the stock x_t and impose the limit condition $\lim_{t\to\infty} x_t = 0$. Imposing symmetric equilibrium on the externality term, and lagging the expression one period, the resulting dynamic first order condition is

$$u_t = \lambda_0 u_{t-1}^{1+\theta} \frac{p_t^X}{p_{t-1}^X}$$

Capital. We generalize this idea and introduce the utilization variable for both types of capital by adding the relationship

$$u_t = u_{t-1}^{\lambda} \left(\frac{P_t^K}{u_t} \frac{u_{t+1}}{P_{t+1}^K} \right)^{\eta} \left(\frac{P_{t-1}^K}{u_{t-1}} \frac{u_t}{P_t^K} \right)^{-\eta}$$

Labor. In the case of labor the expression for utilization, which in the case of labor is closer to the idea of effort - looks slightly different because the nature of the labor input is different. The object L contains many components describing technology and productivity, and utilization factors all components including of course the number of workers. And for convenience L contains also utilization. We therefore make use of the term P^L which is the marginal product $P_t \partial Q_t / \partial L$. In the explicit utilization model above the resulting first order condition would be

$$P_t^L \frac{\partial L_t}{\partial u_t} \frac{\partial u_t}{\partial x_t} + \beta P_{t+1}^L \frac{\partial L_{t+1}}{\partial u_{t+1}} \frac{\partial u_{t+1}}{\partial x_t} = 0$$

and using the fact that $X = L/u$ and $\partial L_t / \partial u_t = L/u$ we get

$$P_t^L \frac{L_t}{u_t} \frac{1}{\frac{L_t}{u_t}} - \beta P_{t+1}^L \frac{L_{t+1}}{u_{t+1}} \frac{1}{\frac{L_{t+1}}{u_{t+1}}} \lambda_0 \frac{\bar{u}_t^{\theta}}{\beta} = 0$$

which becomes

$$P_t^L = P_{t+1}^L \lambda_0 u_t^{\theta}$$

so that the generalized expression used in the model is

$$u_t = u_{t-1}^{\lambda} \left(\frac{P_t^L}{P_{t+1}^L} \right)^{\eta} \left(\frac{P_{t-1}^L}{P_t^L} \right)^{-\eta}$$

3.6.4 Leontief

Here we explain here how the method used to solve an optimization problem with a CES production function also applies to the limit case of zero elasticity. We first solve the general CES problem, and then transform the two-input Leontief problem into a single-input linear technology problem to derive the same optimality conditions as we obtain for the limit of the CES problem.

CES Problem. Consider the CES function of two inputs with profit expression

$$\pi = p_t \underbrace{\left[\sum_{i=1}^{2} (\mu_i)^{\frac{1}{E}} (u_{i,t}X_{i,t})^{\frac{E-1}{E}}\right]^{\frac{E}{E-1}}}_{Q} - \sum_{i=1}^{2}\left(w_{i,t}X_{i,t} + \frac{\gamma_i}{1+\theta}X_{i,t}^{1+\theta}\right)$$

with $\theta > 0$ and elasticity $E > 0$. The price p of output Q is taken as given. We also have an extra variable u which will enter an extension of the problem below.

The first order conditions for X allow us to define the user cost variables p_i

$$p_t\frac{\partial Q_t}{\partial X_{i,t}} = p_t\left(\frac{\mu_i Q_t}{u_{i,t}X_{i,t}}\right)^{\frac{1}{E}} u_{i,t} = p_{i,t} = w_{i,t} + \gamma_i X_{i,t}^{\theta}$$

Now multiply these conditions by X_i

$$p_t\left(\frac{\mu_i Q_t}{u_{i,t}X_{i,t}}\right)^{\frac{1}{E}} u_{i,t}X_{i,t} = p_{i,t}X_{i,t}$$

and sum them to obtain a "budget constraint" expression:

$$p_t Q_t^{\frac{1}{E}}\left[\sum_i \mu_i^{\frac{1}{E}} (u_{i,t}X_{i,t})^{\frac{E-1}{E}}\right] \equiv p_t Q_t = \sum_i p_{i,t}X_{i,t}$$

This expression is important. Before we get back to it, we note that when we solve the problem we invert the first order conditions so we transform them into demand functions

$$u_{i,t}^{1-E}X_{i,t} = \mu_i Q_t\left(\frac{p_{i,t}}{p_t}\right)^{-E}$$

because directly as first order conditions they do not apply to the Leontief problem (as it does not have partial derivatives), but when inverted into demand functions they yield in the limit

$$u_{i,t}X_{i,t} = \mu_i Q_t$$

and in this format these expressions are also part of the solution to the Leontief problem.

Leontief Problem. We can now solve the Leontief problem directly to show we come to the same solution as above. In the Leontief problem we impose that we are always at the kink, $u_1X_1/\mu_1 = u_2X_2/\mu_2$. We then write output in terms of X_1 to obtain the profit expression

$$\pi = p_t\frac{u_{1,t}X_{1,t}}{\mu_1} - w_{1,t}X_{1,t} - \frac{\gamma_1}{1+\theta}X_{1,t}^{1+\theta} - w_{2,t}\frac{u_{1,t}X_{1,t}}{\mu_1}\frac{\mu_2}{u_{2,t}} - \frac{\gamma_2}{1+\theta}\left(\frac{u_{1,t}X_{1,t}}{\mu_1}\frac{\mu_2}{u_{2,t}}\right)^{1+\theta}$$

Now we take the first order condition with respect to X_1

$$0 = p_t \frac{u_{1,t}}{\mu_1} - p_{1,t} - w_{2,t} \frac{u_{1,t}}{\mu_1} \frac{\mu_2}{u_{2,t}} - \gamma_2 \left(\frac{u_{1,t}}{\mu_1} \frac{\mu_2}{u_{2,t}} \right)^{1+\theta} (X_{1,t})^{\theta}$$

and we multiply this condition by X_1 and manipulate to obtain

$$0 = p_t Q_t - p_{1,t} X_{1,t} - p_{2,t} X_{2,t}$$

which is exactly the "budget constraint" we obtained above in the general CES problem. The user cost prices $p_i$are identical and output is defined as it must whether it is a CES or a Leontief function.

Note now that our last step in the CES problem was to show that the limit of the CES demand function

$$u_{i,t}^{1-E} X_{i,t} = \mu_i Q_t \left(\frac{p_{i,t}}{p_t} \right)^{-E}$$

yields the Leontief expression

$$u_{i,t} X_{i,t} = \mu_i Q_t$$

and these two expressions (for $i = 1, 2$) are the ones used above to formulate the Leontief problem as a linear problem at the kink.

So, having defined the p_i, having rewritten the first order conditions as demand functions, and imposing the "budget constraint"

$$0 = p_t Q_t - p_{1,t} X_{1,t} - p_{2,t} X_{2,t}$$

we can solve both the general CES and the limit Leontief cases.

The "Budget Constraint". In this appendix the equation

$$0 = p_t Q_t - p_{1,t} X_{1,t} - p_{2,t} X_{2,t}$$

is defined taking the output price as given and therefore implicitly solves for Q. However, the exact same problem will solve for an endogenous CES price $p_t\left(p_{1,t}, p_{2,t}\right)$ when the problem is embedded in a CES tree where the quantity Q is determined in the above branches of the overall problem.

3.6.5 Data and calibration

Materials and labor inputs as well as stocks of machinery and buildings are matched to national accounts data from ADAM's databank by calibrating the relevant share parameters in the CES structure. The share parameters of the Q^{KL} and Q^{KLB} functions are calibrated so their CES prices, P^{KL} and P^{KLB}, are matched to Paasche chain index prices used in the national accounts. The correction parameters $f^{KLBR}_{sp,t}, etc$ in the Q^{KL}, Q^{KLB}, and Q^{KLBR} objects are not identified and could be set to one. Instead they are calibrated so the share parameters in each CES nest sum to 1. This is helpful if a model user wants to change the share parameters, and otherwise innocuous.

The labor input is imputed using national accounts data from ADAM's databank. The nominal labor input in each sector is measured as the wage sum (total nominal wages paid) plus the imputed total wages of the self-employed. The number of hours worked by both employees and self employed can implicitly be found in ADAM's databank. The nominal labor input of the self-employed is imputed by assuming they have the same hourly wage as employees.

Also, labor is measured in efficiency units. The quantity of efficient labor is found by dividing the nominal input (total wages paid) by the wage index of industrial workers. The interpretation of this way of measuring the labor input is "the amount of labor an industrial worker would deliver for 1 DKK in the base year".

The user cost of capital is a forward looking object, due to the forward looking nature of Tobin's q, and this implies the calibration of parameters in the production and adjustment cost functions depends on the future path of the model. This poses a problem when we are using existing data for calibration as observed input and output prices fluctuate significantly in several sectors. In a perfect foresight environment such as MAKRO, the user cost calculated with these realized values is sometimes negative, and the model cannot solve with a negative user cost. In order to sidestep this problem, when we calibrate the model to fit the historical period, the realized future values of investment prices and depreciation rates are replaced with HP-filtered values.[58,59] Currently, in the historical period installation costs are set to zero, and utilization is fixed at 1 for all periods.

In other parts of the model share parameters are statically calibrated and projected with ARIMA processes, as a trend is sometimes present. The share parameters of the production function are in the historical period affected by the need to fit observed data. Currently we use the ARIMA procedure to forecast the share parameters, based on the static user-cost measures.

[58] The filtered series are proxies for taking expectations, something which cannot be calculated within the model as it is a perfect foresight model.

[59] We derive depreciation rates based on Statistics Denmark data. As they are calculated from nominal investment data using chain index prices and these move over time, this variation must be controlled for. There are also composition effects that create variation in δ. Some capital stocks were for example greatly affected by the seasonal year 2000 storm.

3.6.6 The long run impact of interest rates

One fundamental topic is the effect changes in interest rates have on the investment decision. It is useful here to look at a "steady state" of the first order conditions.

$$(1-\tau)\left[P^Y\frac{\partial Y}{\partial K}-\tau^K P^I\right]=q\left(r+\delta\right)-\left[r-r^D+r^D\tau\right]\mu^D P^I$$

$$q=P^I\left(1-q^{Tax}\right)\equiv P^I\left(1-\frac{\tau\delta^{Tax}}{\left(r+\delta^{Tax}\right)}\right)$$

We put things together and reorganize to focus on the important part on the right hand side

$$RHS\equiv P^I\left(1-q^{Tax}\right)\left(r+\delta\right)-\left[r-r^D+r^D\tau\right]\mu^D P^I$$

$$\Omega\equiv\frac{RHS}{P^I}-\left(1-q^{Tax}\right)\delta\equiv\left(1-q^{Tax}-\mu^D\right)r+\left(1-\tau\right)r^D\mu^D$$

Now decompose the equity rate into the bond rate plus the risk premium, $r=r^D+r^p$ and look at this again:

$$\Omega\equiv r^p\left(1-q^{Tax}-\mu^D\right)+r^D\left(1-q^{Tax}-\tau\mu^D\right)$$

The corporate tax rate is around 0.25, and the debt factor $\mu^D$is 0.4, which implies $q^{Tax}+\tau\mu^D<q^{Tax}+\mu^D<1$.

Nullifying the long run impact of changes in interest rates is done by allowing the long run risk premium to adjust so that long term investment and capital stock are unaffected. We have

$$\frac{RHS}{P^I}\equiv\left(1-q^{Tax}\right)\delta+r^p\left(1-q^{Tax}-\mu^D\right)+r^D\left(1-q^{Tax}-\tau\mu^D\right)$$

with

$$q^{Tax}=\frac{\tau\delta^{Tax}}{\left(r^D+r^p+\delta^{Tax}\right)}$$

We need to allow the risk premium to move to counteract any changes in the bond rate, such that the value of this expression is constant.

3.7 Expressions in the code

We have data that allows us to calculate capital stocks and prices at the sectoral level as in the model. This means that almost all the data constructions necessary for the endogenous part of our firm are done at the right level of disaggregation. Some objects, however, need to be computed from aggregate data, while other objects such financial assets are only relevant in the aggregate. In the expressions that follow the names of variables are written as they are in the code and where possible the corresponding name in the text is included.

3.7.1 Data availability

The value of the firm is matched to the equity value from ADAM's databank. This is the aggregate stock value of all firms in the economy.

The corporate tax rate is calculated using aggregate data. The available aggregate data on tax revenues and tax base does not yield the legal 22% corporate tax rate, and therefore we use the resulting effective corporate tax rate and apply it to all sectors.

A variable that cannot be calibrated on the basis of the data is the debt financing share of the firm, $\mu_t^{Debt} = r_t^{Laan2K}$. We only have the aggregate amount of real estate bonds issued by firms and the aggregate net bond position of all firms (therefore we cannot isolate total corporate bonds issued). The same goes for bank savings and deposits where we only have the net amount. In DREAM the debt financing rate is set to 0,6.[60] In ADAM it is set to 0.5 according to data on consolidated lending relative to issued shares described in the paper "Usercost med egenfinansiering" by Nina Gustafson and Dan Knudsen (2014). With updated series this share is closer to 0.4. The document SKAT (2003) "Den danske selskabsskat - satsreduktion og baseudvidelse" uses the value 0.35 referring to the 2002 report from the EU Commission "Company taxation in the internal market". Finally, in "Vækstplan DK" from 2013 a debt financing rate of 0.4 is used. We follow this and set it to 0.4.

3.7.2 EBITDA

We calculate EBITDA (Earnings Before Interests, Taxes, Depreciations and Amortizations) as

$$EBITDA_{sp,t} = P_{sp,t}^Y Y_{sp,t} - w_t^h L_{sp,t} - P_{sp,t}^R R_{sp,t} - T_{sp,t}^{NetYAfg}$$

where $P_{sp,t}^Y Y_{sp,t}$ is the total value of production taken from national income and product accounts (Nationalregnskabet), $w_t^h L_{sp,t}$ are total wage costs (including the self employed), and $P_{sp,t}^R R_{sp,t}$ are expenses on material inputs, and finally net production taxes.

Net production taxes $T_{sp,t}^{NetYAfg}$ are given by:

$$\begin{aligned}\underbrace{T_{sp,t}^{NetYAfg}}_{T_t^{NP}} = \underbrace{t_{iB,sp,t}^K P_{iB,sp,t}^I K_{iB,sp,t-1}}_{T_{sp,t}^{Grund}=T_t^{Land}} + \underbrace{t_{Im,sp,t}^K P_{Im,sp,t}^I K_{Im,sp,t-1}}_{T_{sp,t}^{VirkVaegt}=T_t^{Weight}} + \\ + \underbrace{t_{sp,t}^L w_t^h \left(1 - r_{sp,t}^{Selvst}\right) L_{sp,t}}_{T_{sp,t}^{NetLoenAfg}=T_t^{Payroll}} + \underbrace{T_{sp,t}^{NetYAfgRest}}_{T_t^{Rest}}\end{aligned}$$

where $T_{sp,t}^{Grund}$ is the total nominal land tax revenue (grundskyld), $T_{sp,t}^{VirkVaegt}$ is the total nominal vehicle weight tax revenue (vægtafgift), $T_{sp,t}^{NetLoenAfg}$ is the payroll tax (total

[60]The reference supporting 0.6 in DREAM, "Schultz Møller (1993)", could not be found.

nominal tax revenue on the wage sum), and $T_{sp,t}^{NetYAfgResidual}$ are other net production taxes.

The respective tax rates used in the model are implicitly calculated on the basis of tax revenues. They are the land tax rate $t_{iB,sp,t}^{K}$, the weight tax rate $t_{Im,sp,t}^{K}$, the payroll tax rate $t_{sp,t}^{L}$, and we obtain directly from the data the share of the labor input from the self employed in each sector, $r_{sp,t}^{Selvst}$.

In the housing sector the rental value of owned houses is subtracted, as this is included in the national accounts determination of capital income, but does not belong to firms.

3.7.3 EBT

EBT (Earnings Before Taxes) is calculated using a disaggregated tax-related capital stock $K_{k,sp,t-1}^{Skat}$. We assume the same debt ratio for all sectors so that we can calculate disaggregated loans/debt as we have data on capital stocks. The only thing we do not have disaggregated by sector is the net interest earned on their bank deposits and bond assets. We assume that these are proportional to the capital stock of the sector.

Earnings before taxes are given by:

$$EBT_{sp,t} = EBITDA_{sp,t} - \sum_{k} \delta_{k,t}^{Skat} K_{k,sp,t-1}^{Skat}$$

$$+ \left[\underbrace{r_{bank,t}^{Rente} V_{bank,t-1}^{Virk}}_{\text{Assets, Net Deposits.}} + r_{obl,t}^{Rente} V_{obl,t-1}^{Virk} - r_{RealKred,t}^{Rente} V_{RealKred,t-1}^{Virk} \right] \frac{V_{sp,t-1}^{VirkK}}{V_{t-1}^{VirkK}}$$

and adding and subtracting the implicit corporate debt object this can be rewritten as:

$$EBT_{sp,t} = EBITDA_{sp,t} - \sum_{k} \delta_{k,t}^{Skat} K_{k,sp,t-1}^{Skat}$$

$$+ \left[\underbrace{r_{bank,t}^{Rente} V_{bank,t-1}^{Virk} + r_{obl,t}^{Rente} V_{obl,t-1}^{Virk} + r_{obl,t}^{Rente} r_{t-1}^{OblLaan2K} V_{t-1}^{VirkK}}_{\text{Net bank deposits plus bonds as assets, } r_t^B A_{t-1}^B \text{, exogenous to optimization}} \right] \frac{V_{sp,t-1}^{VirkK}}{V_{t-1}^{VirkK}}$$

$$- \left[\underbrace{r_{obl,t}^{Rente} r_{t-1}^{OblLaan2K} V_{t-1}^{VirkK} + r_{RealKred,t}^{Rente} V_{RealKred,t-1}^{Virk}}_{\text{Bonds as liabilities, } r_{obl,t}^{Rente} r_{t-1}^{Laan2K} V_{t-1}^{VirkK} = r_t^D D_{t-1} = r_t^D \mu_t^D P_t^I K_{t-1} \text{, endogenous.}} \right] \frac{V_{sp,t-1}^{VirkK}}{V_{t-1}^{VirkK}}$$

since $V_{obl,t-1}^{Virk}$ has been defined as a net quantity (bonds as assets minus imputed corporate bond liabilities, the imputation is detailed below). Note that shares as assets are left out of earnings altogether as they are not subject to taxation.

Capital stock

The value of the aggregate productive capital stock is denoted V_t^{VirkK} and given by

$$V_t^{VirkK} = \sum_{k} \sum_{sp} P_{k,sp,t}^{I} K_{k,sp,t} - P_{iB,Bol,t}^{I} K_t^{Bolig}$$

It is the sum over both machinery and building capital of all private sectors excluding privately owned housing K_t^{Bolig}. This correction for housing applies only to the housing sector. For all sectors other than housing we can write

$$V_{sp,t}^{VirkK} = \sum_k P_{k,sp,t}^I K_{k,sp,t}$$

where we note again that the investment price is a CES object as investments into capital are the result of purchases across all sectors, domestic and foreign.

In the housing sector (which only has buildings and does not have machinery) we have

$$V_{Bol,t}^{VirkK} = P_{iB,Bol,t}^I K_{iB,Bol,t} - P_{Bol,t}^I K_t^{Bolig}$$

In the code the object $K_{iB,Bol,t}$ is the total stock of housing, both rental and owned, and

$$K_{iB,Bol,t} - K_t^{Bolig}$$

is the stock of rental housing, which makes K_t^{Bolig} the stock of owned housing (which is determined endogenously in the household problem).

Book/tax capital

Here $\delta_{k,t}^{Skat}$ is the depreciation rate considered for tax deduction purposes, and $K_{k,sp,t}^{Skat}$ is the book/tax value of capital stock of type k in sector *sp*. The firm benefits from a favorable tax treatment of capital depreciation with a tax deductible depreciation rate, $\delta_{k,t}^{Skat}$, which can be higher than the actual rate of depreciation. Therefore there is a nominal aggregate which accumulates and is the source of the tax benefit. We call it the "tax value" of capital and it is given by:

$$K_{Im,t}^{Skat} = \left(1 - \delta_{Im,t}^{Skat}\right) K_{Im,t-1}^{Skat} + \sum_{sp} P_{Im,sp,t}^I I_{Im,sp,t}$$

$$K_{iB,t}^{Skat} = \left(1 - \delta_{iB,t}^{Skat}\right) K_{iB,t-1}^{Skat} + \sum_{sp} P_{iB,sp,t}^I I_{iB,sp,t} - P_{iB,Bol,t}^I I_{Tot,t}^{Bolig}$$

where $P_{k,sp,t}^I$ is the investment price on type k capital in private sector sp, and where Im is machinery capital, iB is building capital and Bol is the housing sector. This price results from CES cost minimization and combines prices of imported investment goods and of investment goods from the different domestic sectors. Investment $I_{k,sp,t}$ is the corresponding CES quantity aggregate. Total household housing purchases, $P_{iB,Bol,t}^I I_{Tot,t}^{Bolig}$, must be subtracted from the firms tax value of (structures) building capital, as the data bundles together residential and industrial buildings.

Financial objects, $V_{bank,t-1}^{Virk}$, $V_{obl,t-1}^{Virk}$, $V_{RealKred,t-1}^{Virk}$

In our model description above we separated firm assets into two broad types, stocks A^S which do not pay taxes on their income, and bonds A^B which do. Both these objects are a stylized description of several objects inside the firm. In data and code terminology Assets (akt) consist of bonds (obl), domestic stocks (IndlAktier), foreign stocks (UdlAktier), bank deposits (Bank) and gold (Guld).

Domestic and foreign stocks held are left out of the EBT expression. Dividends received are not subject to corporate tax, as the firms paying out the dividends have already paid it. Since the firms holding the asset do not pay taxes on these returns we can completely separate the value of stocks from the problem of the firm and therefore these assets do not appear here. Neither does gold which is an asset that pays no dividend and is valued at its face/market value.

Bonds held by the firm as a claim on other agents include real estate bonds issued both by other firms and by households. Interest rates on these assets are exogenous.[61] Bonds issued by the firm are a liability (RealKredit for mortgage bonds, and Obligationer for corporate debt).

The objects $V^{Virk}_{bank,t-1}$, $V^{Virk}_{obl,t-1}$, are the face values of aggregate bank deposits and net bond holdings - bonds held as assets minus corporate bonds issued - so that mortgages issued by the firm are excluded. The object $V^{Virk}_{RealKred,t-1}$ is the aggregate of the face value of mortgages issued by firms (Realkredit). We do *not* have this variable disaggregated by sectors. Mortgages of other agents held as assets are included in $V^{Virk}_{obl,t-1}$, and make up more than half of all bonds held by firms.[62] In the model bonds held as assets are kept constant and grow with the exogenous long run growth trend.

In the model all agents earn the same interest, dividend and revaluation rates when holding the same assets. In the data this is not true as some bank debt is written off and portfolio composition details vary. For example, not all bonds are identical in the data and different agents hold them in different proportions, but this is at a disaggregation level below that modeled.[63]

Debt

In order to match the EBT expression with the one in the model we need to understand how corporate debt is imputed. Total debt in each sector is calculated by taking the data on investment prices and respective capital stocks for buildings and machinery and multiplying this value by the factor $\mu^D_t = r^{Laan2K}_t = 0.4$. For all sectors except housing we have

$$D_{sp,t} \equiv r^{Laan2K}_t \left(P^I_{b,sp,t} K_{b,sp,t} + P^I_{m,sp,t} K_{m,sp,t} \right)$$

while in the housing sector we have

$$D_{Bol,t} \equiv r^{Laan2K}_t \left(P^I_{b,Bol,t} K_{b,Bol,t} - P^I_{b,Bol,t} K^{Bolig}_t \right)$$

Given aggregate data on corporate mortgages, $V^{Virk}_{RealKred,t}$, we can impute aggregate corporate debt as the difference

$$r^{OblLaan2K}_t V^{VirkK}_t = \sum_{sp} D_{sp,t} - V^{Virk}_{RealKred,t}$$

which implies

$$V^{Virk}_{RealKred,t} \equiv \underbrace{r^{RealKredLaan2K}_t}_{\mu^{mortgages}} V^{VirkK}_t = \left(\underbrace{r^{Laan2K}_t}_{\mu^D_t = 0.4} - \underbrace{r^{OblLaan2K}_t}_{\mu^{corporate}} \right) V^{VirkK}_t$$

[61] Dividends on foreign stocks are exogenous. Capital gains (revaluation rates) are also exogenous except for those on domestic equity. The revaluation (Omvurdering) of bonds is set to zero in the forecast. Both historically and in the forecast there is no revaluation of bank deposits or gold.

[62] Mortgage bonds are assets widely held by investment vehicles which are strongly represented in the service sector of our model.

[63] The code contains adjustment terms that make total returns and net interest income match. It is an artificial lump-sum transfer that makes bookkeeping consistent. The households, the public sector and the foreign sector have equivalent adjustment terms and they sum to zero. In the forecast they are all set to 0.

3.7.4 Corporate taxes

The aggregate of corporate tax revenue is given by the sum of regular corporate tax and the additional tax on the extraction sector:

$$T_t^{Virk} = f_t^{Selskab} \tau_t^{Selskab} EBT_t + \underbrace{\tau_t^{SelskabNord} EBT_{udv,t}}_{\text{Oil sector}}$$

where $f_t^{Selskab}$ is a factor capturing the difference between the statuary 22 pct. tax rate and the effective rate calculated from tax revenue and tax base. The rate $t_t^{SelskabNord}$ is an implicit tax rate on the proceeds from oil extraction in the North Sea.

We emphasize again that this expression for corporate tax revenues T_t^{Virk}, denotes the aggregate economy tax revenue. The subcomponent of the north sea oil only applies to that sector. The statuary tax rate is treated as the marginal tax in firm decisions such that $\tau_{sp,t} = \tau_t^{Selvskab}$ in all sectors except the extraction sector.

3.7.5 FCF

$FCT_{sp,t}$ is the Free Cash Flow of the firm, excluding income from financial assets.

$$\begin{aligned} FCF_{sp,t} = {} & EBITDA_{sp,t} - T_{sp,t}^{Selskab} - \underbrace{V_{sp,t}^{VirkI}}_{P_{sp,t}^{I} I_{sp,t}} \\ & - \left(1 + r_{obl,t}^{Rente}\right) r_{t-1}^{Laan2K} V_{sp,t-1}^{VirkK} + r_t^{Laan2K} V_{sp,t}^{VirkK} \\ & + \underbrace{TilVirk_t^{NetOvf}}_{T_t^0} \frac{FCF_{sp,t}}{FCF_t} \end{aligned}$$

Again, shares as assets in the firm are excluded. So are bonds as assets. However, taxes on the income generated by bond holdings are implicit inside total corporate tax revenues and we separate them below. We assume that other net transfers, $TilVirk_t^{NetOvf}$, are split proportionally to the capital stocks of the sectors.

Net transfers

are given by:

$$\underbrace{TilVirk_t^{NetOvf}}_{T_t^0} = \begin{cases} TilVirk_t^{Off} - FraVirk_t^{Off} - FraVirk_t^{Hh} + \\ \\ +JordKoeb_t^{Off} - SelvstKapInd_t + IndRest_t^{Virk} \end{cases}$$

The different items in this object are: capital transfers from the public sector to firms, $TilVirk_t^{Off}$, capital transfers from firms to the public sector, $FraVirk_t^{Off}$, capital transfers from firms to households $FraVirk_t^{Hh}$, public land purchases $JordKoeb_t^{Off}$ (government buying land from firms), capital income transfers directly from firms to households $SelvstKapInd_t$ (profits from individually owned firms that are deducted from the sector aggregate), and finally net capital transfers to firms from abroad $IndRest_t^{Virk}$.[64]

[64]In ADAM $IndRest_t^{Virk}$ is the firm's net lending (nettofordringserhvervelse) residual, so its contents are not explicit.

Investment

The value of aggregate investment expenditure is, in the data, the sum of total private investments, V_t^{VirkI}. In order to obtain total investment expenditure by firms we must exclude household investments as these are being accounted for as expenses in the budget constraint of the household:

$$V_t^{VirkI} = \sum_k \sum_{sp} P_{k,sp,t}^I I_{k,sp,t} - \left(P_{iB,Bol,t}^I I_{Tot,t}^{Bolig} + I_t^{Hhx} \right)$$

where $P_{iB,Bol,t}^I I_{Tot,t}^{Bolig}$ is the investment in buildings by households and I_t^{Hhx} measures the aggregate value of household non-housing investments (these are mainly investments in capital, such as buying tools, by self employed workers).

Note that implicit in the investment sum

$$\sum_k \sum_{sp} P_{k,sp,t}^I I_{k,sp,t}$$

is the sectoral disaggregation of what we are removing,

$$P_{iB,Bol,t}^I I_{Tot,t}^{Bolig} + I_t^{Hhx}$$

In particular, the housing term pertains only to the housing sector, so that, although we do not have a sectoral disaggregation of I_t^{Hhx}, if we did we would write for all sectors except housing

$$V_{sp,t}^{VirkI} = \sum_k P_{k,sp,t}^I I_{k,sp,t} - I_{sp,t}^{Hhx}$$

while for the housing sector we would write

$$V_{Bol,t}^{VirkI} = P_{iB,Bol,t}^I I_{iB,Bol,t} - P_{iB,Bol,t}^I I_{Tot,t}^{Bolig} - I_{Bol,t}^{Hhx}$$

As it is, because I_t^{Hhx} is an aggregate expense incurred in the household problem, here it is treated as an exogenous object which can be subtracted from the aggregate budget constraint of all firms (so that we do not account for this expense twice) and separated from the problem as we do for financial assets.

3.7.6 Matching the value of the firm

Until now we have detailed how we account for the endogenous and exogenous parts of firm value. The endogenous part is built from data on input quantities and prices for which we do not always have complete data in the historical period. However we have face/market values for the objects in the exogenous part, as well as data for the total value of firms which is the aggregate equity value from ADAMs databank. This allows us to compute the endogenous component as a residual when needed in the historical period.

Going forward in time the model solves endogenously for inputs, outputs, and prices, and the exogenous part enters as an exogenous forecast. The reason this block of assets remains exogenous is that there is separation between the portfolio held by firms and the portfolios held by households and pension firms, and this separation stems from the fact that this is an open economy without financial frictions.

We could therefore ignore the exogenous block altogether, but for the fact that some parts of it enter the government budget constraint either as tax revenues or transfers. And tax revenues on the returns of some of these assets must then be forecast. This implies forecasting these components of firm wealth for given tax rates and the details of this procedure will appear here.

3.8 Endogenous operational surplus

In the code we do not have Lagrange multipliers attached to the expressions for operational surplus. We list here what can be found and what to look for.

3.8.1 All sectors except housing

For all sectors except housing these are the variable components of profits:

$$y_t^{Endo} = (1-\tau_t)\left[P_t^Y Y_t - \hat{w}_t L_t - P_t^R R_t - \tau_t^K P_t^I K_{t-1} - T_t^{Rest}\right]$$

$$-P_t^I I_t + \tau_t \delta_t^{Tax} K_{t-1}^{Tax}$$

$$-(1-\tau_t)\, r_t^D D_{t-1}$$

$$+D_t - D_{t-1}$$

and in the code this equation is:

$$\begin{aligned}
\pi_{sp,t}^{Var} \equiv & \left(1 - f_t^{Selskab} t_t^{Selskab}\right) EBITDA_{sp,t} \\
& - \textstyle\sum_k P_{k,sp,t}^I I_{k,sp,t} + f_t^{Selskab} t_t^{Selskab} \sum_k \delta_{k,t}^{Skat} K_{k,sp,t-1}^{Skat} \\
& - r_{Obl,t}^{Rente}\left(1 - f_t^{Selskab} t_t^{Selskab}\right) r_{t-1}^{Laan2K} \textstyle\sum_k P_{k,sp,t-1}^I K_{k,sp,t-1} \\
& + \left(r_t^{Laan2K} \textstyle\sum_k P_{k,sp,t}^I K_{k,sp,t} - r_{t-1}^{Laan2K} \sum_k P_{k,sp,t-1}^I K_{k,sp,t-1}\right)
\end{aligned}$$

3.8.2 The housing sector

Model

Operational surplus

$$y_t^{Endo} = (1-\tau_t)\left[\underbrace{\left[P_t^Y \phi_k - P_t^R \phi_R - \hat{w}_t \phi_l - \tau_t^K P_t^I\right] K_{t-1} - T_t^{Rest}}_{\text{EBITDA}}\right] + T_t^0$$

$$+\mu_t^D P_t^I K_t$$

$$-\left(1 + r_t^D (1-\tau_t)\right) \mu_{t-1}^D P_{t-1}^I K_{t-1}$$

$$-P_t^I I_t + \tau_t \delta_t^{Tax} K_{t-1}^{Tax}$$

Code

In the code the object $K_{b,Bol,t}$ is the total stock of housing, both rental and owned, and

$$\left(K_{b,Bol,t} - K_t^{Bolig}\right)$$

is the stock of rental housing, which makes K_t^{Bolig} the stock of owned housing. Operational surplus (which only includes rental housing) is then:

$$\pi_{Bol,t}^{Var} = \left(1 - f_t^{Selskab} t_t^{Selskab}\right) EBITDA_{Bol,t} + TilVirk_t^{NetOvf}$$

$$+ r_t^{Laan2K} P_{b,Bol,t}^{I} \left(K_{b,Bol,t} - K_t^{Bolig}\right)$$

$$- \left(1 + r_{Obl,t}^{Rente} \left(1 - f_t^{Selskab} t_t^{Selskab}\right)\right) r_{t-1}^{Laan2K} P_{b,Bol,t-1}^{I} \left(K_{b,Bol,t-1} - K_{t-1}^{Bolig}\right)$$

$$- P_{b,Bol,t}^{I} \left(I_{b,Bol,t} - I_{Tot,t}^{Bolig}\right) + f_t^{Selskab} t_t^{Selskab} \delta_{b,t}^{Skat} K_{b,Bol,t-1}^{Skat}$$

where investments and the capital stock are corrected for privately owned houses. Also, the EBITDA is only for the rental part, as the rental value of owner housing is subtracted.

3.9 Firm Investment with Costly External Finance

The simplest model of the firm abstracts from costly external finance. The firm has free access to funds at an exogenous rate r_t which is the discount rate of its cash flows, and which is the required rate of return demanded by all investors.[65] A simple illustration of this model is the one variable example where the capital stock obeys the standard law of motion $K_t = (1-\delta) K_{t-1} + I_t$, and profits are given by revenues minus investment costs, $\pi_t = Y(K_{t-1}) - P_t^I I_t$. The unconstrained optimal choice of capital obeys

$$\frac{\partial \pi_t}{\partial K_t} + \beta_{t+1} \frac{\partial \pi_{t+1}}{\partial K_t} = 0$$

where $\beta_{t+1} = 1/(1+r_{t+1})$ is the discount factor which is exogenous to the firm and is not affected by how much the firm chooses to invest.

This is, however, an incomplete model since financing the activity of the firm is a complex process. When firms finance their activity the immediate source of funds is retained earnings or internal finance. This is often not enough and firms also interact with banks, issue corporate debt, and raise or buy back equity. One common characteristic of all sources of outside finance is that it is costly and therefore we want to extend the model of the firm by defining this cost. Before that, in order for costly external finance to matter the firm must be in need of funds. It must be constrained. The external finance constraint must bind.

The simplest model of costly outside finance adds a constraint $\pi_t \geq \bar{\pi}$. Outside finance beyond this threshold is infinitely costly. With a Lagrange multiplier ξ the optimal decision obeys

$$\frac{\partial \pi_t}{\partial K_t} + \left[\beta_{t+1} \frac{1+\xi_{t+1}}{1+\xi_t}\right] \frac{\partial \pi_{t+1}}{\partial K_t} = 0$$

where if $\pi_t < \bar{\pi}$ then $\xi_t > 0$ (and large enough). The firm can raise cash freely up until the threshold and after that the cost of outside finance is prohibitive.

The optimality condition shows what will be a very useful characteristic as we change the model: this financial friction is fully captured in the generalized discount factor. While this is not a general property of models of firm finance, we will choose our model so that we have it.

When the constraint binds this generalized discount factor is lower (the future becomes less relevant). This is intuitive as the effect of costly external finance is to *reduce* investment. However, that is not the only effect of costly external finance on the firm. In fact, more interesting over the business cycle is how this constraint moves. Although costly external finance acts to reduce investment, as long as the constraint is normally binding what matters is how the constraint tightens and relaxes. These effects through the constraint help propagate or *accelerate* the business cycle.

In MAKRO we model the costs of external finance assuming that they generally bind. We do this by using a differentiable cost function as in Gomes (2001) as a flexible way of modeling the above Lagrange multiplier. His model is as follows. Dividends are

$$d_t = \pi_t - \xi(m_t)$$

with the function ξ having value zero for $m \leq 0$ and a positive increasing function for $m > 0$ where $m_t = -\pi_t = P_t^I I_t - Y_t$, and it is easy to verify that it yields the same first order condition

$$\frac{\partial \pi_t}{\partial K_t} + \left[\beta_{t+1} \frac{1+\frac{\partial \xi_{t+1}}{\partial m_{t+1}}}{1+\frac{\partial \xi_t}{\partial m_t}}\right] \frac{\partial \pi_{t+1}}{\partial K_t} = 0$$

where the Lagrange multipliers are replaced with the derivatives of the cost function.

[65] There are also no conflicts of interest between management and ownership, and no issues of corporate control.

3.9.1 The Financial Constraint in MAKRO

The firm maximizes

$$V_t = m_t - \xi_t + \beta_{t+1} V_{t+1}$$

where m_t is the free cash flow of the firm (excluding financial assets). The positive valued function $\xi_t \equiv \xi\left(m_t - \overline{m}_t\right) \geq 0$ models the cost of financial frictions. The first order condition with respect to a given input X_t is

$$\frac{\partial m_t}{\partial X_t} + \hat{\beta}_{t+1} \frac{\partial m_{t+1}}{\partial X_t} = 0$$

where the augmented discount factor $\hat{\beta}_{t+1}$ is given by

$$\hat{\beta}_{t+1} = \beta_{t+1} \frac{1 - \partial \xi_{t+1} / \partial m_{t+1}}{1 - \partial \xi_t / \partial m_t}$$

We can rewrite the augmented discount factor as $\hat{\beta}_{t+1} = 1/(1 + r_{t+1} + r^p_{t+1} + r^{\xi}_{t+1})$ by defining and additional risk premium r^{ξ}_{t+1}. We make use of this in the code as we detail below.

We specify the financial friction function ξ as a differentiable symmetric function with derivative[66]

$$\frac{\partial \xi_t}{\partial m_t} \equiv \mu \tanh\left(\kappa\left[m_t - \overline{m}_t\right]\right)$$

where $\lim_{m_t \to \infty} \partial \xi_t / \partial m_t = \mu$ and $\lim_{m_t \to -\infty} \partial \xi_t / \partial m_t = -\mu$.[67] Numerically, with $\mu = 2\%$ the firm pays a 2 pct. fee for outside finance ($m_t < \overline{m}_t$). The parameter $\kappa > 0$ controls the slope of the derivative around $m_t - \overline{m}_t = 0$.

3.9.2 Homogeneity

The function $\xi_{s,t}$ is sector specific with sector subscript s, and is approximately homogeneous if $\overline{m}_{s,t} = 0$ and with high enough κ. To preserve long run homogeneity with a free cash flow target different from zero we let $\overline{m}_{s,t}$ endogenously adjust as

$$\overline{m}_{s,t} = \gamma \overline{m}_{s,t-1} + (1 - \gamma)\, m_{s,t} - \epsilon_{s,t}$$

where γ controls the speed of adjustment and $\epsilon_{s,t}$ can be used to set $\overline{m}_{s,t} = m_{s,t}$ in the baseline forecast. We calibrate $\kappa_{s,t} = \frac{\hat{\kappa}}{K_{s,t}}$ (other measures of sector size can be used) to ensure similar behavior across sectors. For strict homogeneity $\kappa_{s,t}$ would need to endogenously adapt to changes in $K_{s,t}$, but this is unlikely to matter in practice as $m_{s,t} - \overline{m}_{s,t}$ converges to zero.

It is useful to define the 'habit adjusted' free cash flow $\hat{m}_{s,t}$:

$$\hat{m}_{s,t} \equiv m_{s,t} - \overline{m}_{s,t} = \gamma\left[m_{s,t} - m_{s,t-1} + \hat{m}_{s,t-1}\right] + \epsilon_{s,t}$$

Rearranging the augmented discount factor we get the expression in the code for the additional risk premium

$$r^{\xi}_{s,t+1} = \left[1 + r_{t+1} + r^p_{t+1}\right] \left[\frac{1 - \mu_s \tanh\left(\kappa_s \hat{m}_{s,t}\right)}{1 - \mu_s \tanh\left(\kappa_s \hat{m}_{s,t+1}\right)} - 1\right]$$

[66] The derivative is modeled using the hyperbolic tangent function $tanh(x) = \left(e^x - e^{-x}\right) / \left(e^x + e^{-x}\right)$. The derivative is well defined for the entire support of x which makes the primitive function ξ two sided rather than a one sided penalty function for negative profits. The function ξ then penalizes the firm for missing profit targets. In a specific application it can be viewed as the cost of issuing ($m_t < \overline{m}_t$) or re-purchasing ($m_t > \overline{m}_t$) equity.

[67] Given this specification for the first derivative, the primitive cost function is approximately the absolute value function $\xi_t \approx \mu \left|m_t - \overline{m}_t\right|$.

3.9.3 Acceleration

The financial constraint model we use is closer to the model in Gomes (2001) than to collateral acceleration models such as Bernanke and Gertler (1989). The factors relaxing the constraint in good times come from the expansion of revenues as prices increase and from the increased production ability as capital accumulates. There is no additional contribution from an improvement in collateral values, or from any additional relaxation of the constraint from a goodwill effect arising from extra revenues.[68] Such features are currently under consideration.

[68] These mechanisms are discussed in recent work. See Lian and Ma (2020) and Drechsel (2021) and references therein.

4 Price setting behavior

The price of goods paid by consumers $P_{s,t}$ is generally not the same as the price which results from production optimization, $P^0_{s,t}$.[69] The optimization price is a construction through a nested sequence of cost minimization problems and reflects production technology and features of input markets. Perfect competition in the market for a particular good implies the two prices are the same:

$$\underbrace{(P_{s,t}|_{\text{all s}}, w_t, r_t)}_{\text{Input Prices}} \underbrace{\Longrightarrow}_{\text{Production}} \underbrace{P^0_{i,t}|_{i\in s}}_{\text{Optimization Price}} \equiv \underbrace{P_{i,t}|_{i\in s}}_{\text{Final Price}}$$

This solution generates price dynamics that are very different from the data in that they respond much faster to shocks and to any changes in the economy. Observed prices are more sluggish than the ones generated by the perfect competition solution. The standard way to solve this problem is to add an intermediate layer of price setting behavior between the producing firm and the consumer so that the final price is not identical to the optimization price.

$$\underbrace{(P_{s,t}|_{\text{all s}}, w_t, r_t)}_{\text{Input Prices}} \underbrace{\Longrightarrow}_{\text{Production}} \underbrace{P^0_{i,t}|_{i\in s}}_{\text{Optimization Price}} \underbrace{\Longrightarrow}_{\text{Price Setting}} \underbrace{P_{i,t}|_{i\in s}}_{\text{Final Price}}$$

This price setting intermediate model is often modeled as a monopolistic competition problem. We also adopt that model and apply it to all private production sectors except housing. This yields positive markups in manufacturing and services. These sectors account for circa 73.5% of all nominal gross private production in 2017. In all other sectors we obtain either zero or negative markups in the data period and we therefore treat these sectors as perfectly competitive from 2018 onwards. The exceptions are extraction and construction and we discuss them below.

In addition to monopolistic competition we have price rigidity coming from adjustment costs of changing prices. Monopolistic competition alone does not generate price rigidity. It merely provides a theoretical foundation for price setting behavior which in turn generates price rigidity.

4.1 Monopolistic competition and price rigidity

Monopolistic competition is a superstructure added to the problem of the firm, where every sector is thought of as having a continuum of firms with unit mass, each producing an individual "variety". Demand for all varieties is a standard CES aggregator with a demand elasticity, and in equilibrium the price paid by the consumer, $P_{s,t}$, is a markup over the marginal cost of production which reflects this elasticity. The equilibrium is symmetric so that in the end the unit mass of firms within a sector looks like a single representative firm.

While the marginal cost price $P^0_{s,t}$ is flexible the consumer price is not, and therefore on top of this structure we add price rigidity through a cost of price adjustment similar to Kravik, Motzfeldt and Mimir (2019).[70]

[69] Consumer/final prices in private production sectors, $P_{s,t}$, are determined in this section. This price $P_{s,t}$ is the price before product taxes (i.e. duties, VAT and customs) are levied. The price index for production in the public sector is described in the public production chapter, and the price of housing is also excluded from this analysis.

[70] Kravik, Erling Motzfeldt og Yasin Mimir (2019). "Navigating with NEMO". I: p. 177.

4.2 Monopolistic Competition Model

In what follows we disregard the sector index s. Within each sector firms are subject to monopolistic competition. In the monopolistic competition set-up all firms within each sector face the same demand elasticity, η_t, and the aggregate price over all firms in a given sector, P_t, is a CES price index.

Without price-adjustment costs P_t would be a markup over the marginal cost of production, P_t^0. However, prices are sticky as we assume these firms pay a quadratic adjustment cost to change them. The adjustment cost function follows Rotemberg (1982), but instead of the cost being applied to changes in the price level, p_t/p_{t-1}, it is applied to changes in inflation which allows for richer dynamics.[71]

The monopolistic competition model generates the following demand aimed at the individual firm:

$$y_t^j = \left(\frac{p_t^j}{P_t}\right)^{-\eta_t} Y_t$$

In the absence of adjustment costs firms would set their price as the following markup over marginal costs:

$$P_t^* = \frac{\eta_t}{\eta_t - 1} P_t^0 = \left(1 + \frac{1}{\eta_t - 1}\right) P_t^0 = (1 + \theta_t) P_t^0$$

In the presence of price adjustment costs the markup relationship is more general.

4.2.1 Optimization Problem

Each firm j in this sector then faces adjustment costs of changing prices given by:

$$g_t^j = \frac{\gamma}{2}\left[\frac{p_t^j/p_{t-1}^j}{p_{t-1}^j/P_{t-2}} - 1\right]^2 P_t Y_t$$

where p_t^j is firm $j's$ chosen price, P_t is the aggregate price level in the sector, and Y_t is the sector's total production. The derivatives of the adjustment cost function (multiplied by the price output ratio) are given by

$$\frac{p_t^j}{y_t^j}\frac{\partial g_t^j}{\partial p_t^j} = \gamma \frac{p_t^j/p_{t-1}^j}{p_{t-1}^j/P_{t-2}}\left[\frac{p_t^j/p_{t-1}^j}{p_{t-1}^j/P_{t-2}} - 1\right]\frac{P_t Y_t}{y_t^j}$$

$$\frac{p_t^j}{y_t^j}\frac{\partial g_{t+1}^j}{\partial p_t^j} = -2\gamma \frac{P_{t-1}p_{t+1}^j}{p_t^j \times p_t^j}\left[\frac{P_{t-1}p_{t+1}^j}{p_t^j \times p_t^j} - 1\right]\frac{P_{t+1} Y_{t+1}}{y_t^j}$$

Firm j in this sector solves the dynamic problem

$$V_t^j = max_{p_t^j}\left\{\left(p_t^j - P_t^{j,0}\right) y_t^j - g_t^j + \beta_{t+1} V_{t+1}^j\right\}$$

subject to

$$y_t^j = \left(\frac{p_t^j}{P_t}\right)^{-\eta_t} Y_t$$

and to the adjustment cost function above.

The first order condition is

[71] [Rotemberg, Julio (1982). "Monopolistic Price Adjustment and Aggregate Output". I: Review of Economic Studies 49.4, s. 517–531.]

$$p_t^j = \frac{\eta_t}{\eta_t - 1} P_t^{j,0} - \frac{1}{\eta_t - 1} \frac{p_t^j}{y_t^j} \left[\frac{\partial g_t^j}{\partial p_t^j} + \beta_{t+1} \frac{\partial g_{t+1}^j}{\partial p_t^j} \right]$$

After some algebra we obtain

$$\begin{aligned} p_t^j &= \frac{\eta_t}{\eta_t - 1} P_t^{j,0} \\ &- \frac{\gamma}{\eta_t - 1} \frac{p_t^j / p_{t-1}^j}{p_{t-1}^j / P_{t-2}} \left[\frac{p_t^j / p_{t-1}^j}{p_{t-1}^j / P_{t-2}} - 1 \right] \frac{P_t Y_t}{y_t^j} \\ &+ \frac{\gamma}{\eta_t - 1} \left(\beta_{t+1} 2 \frac{P_{t-1} p_{t+1}^j}{p_t^j \times p_t^j} \left[\frac{P_{t-1} p_{t+1}^j}{p_t^j \times p_t^j} - 1 \right] \frac{P_{t+1} Y_{t+1}}{y_t^j} \right) \end{aligned}$$

and using symmetry and the unit mass assumption we obtain the final expression

$$\begin{aligned} P_t = {} & (1 + \theta_t) P_t^0 \\ & - \psi_t \left[\frac{P_t / P_{t-1}}{P_{t-1} / P_{t-2}} - 1 \right] \frac{P_t / P_{t-1}}{P_{t-1} / P_{t-2}} P_t \\ & + 2 \beta_{t+1} \psi_t \frac{Y_{t+1}}{Y_t} \left[\frac{P_{t+1} / P_t}{P_t / P_{t-1}} - 1 \right] \frac{P_{t+1} / P_t}{P_t / P_{t-1}} P_{t+1} \end{aligned}$$

where β_{t+1} is the discount factor and $\psi_t \equiv \gamma \theta_t$.

Note that the final markup is given by

$$\begin{aligned} P_t - P_t^0 = {} & \theta_t P_t^0 \\ & - \psi_t \left[\frac{P_t / P_{t-1}}{P_{t-1} / P_{t-2}} - 1 \right] \frac{P_t / P_{t-1}}{P_{t-1} / P_{t-2}} P_t \\ & + 2 \beta_{t+1} \psi_t \frac{Y_{t+1}}{Y_t} \left[\frac{P_{t+1} / P_t}{P_t / P_{t-1}} - 1 \right] \frac{P_{t+1} / P_t}{P_t / P_{t-1}} P_{t+1} \end{aligned}$$

which can be very close to zero even with a significant degree of monopolistic market power.

4.3 Performance and discussion

The price setting model is a filter which takes as an input the optimization price which is highly volatile, and adds structure to it, generating a consumer price object which behaves in a more sluggish way.

When we take the price setting model to the data we assume independence of θ and ψ. We obtain a positive value of θ in manufacturing (4%), services (9%), and extraction (29%). In agriculture, energy, and sea transport θ is volatile and often negative during the data years, and negative in 2017, and is therefore set to zero going forward. In these sectors we use the perfect competition environment where $P^0 = P$ so that both θ and ψ vanish.

The extraction sector has a positive markup but it is hard to think of its price as being determined anywhere else than in the world market. Therefore, despite the large positive markups this is not a sector where the price setting structure is likely to apply, and we exogenize its price instead. In the construction sector we obtain use a zero markup, $P_t = P_t^0$, with positive adjustment costs.

Manufacturing and services are the two largest private sectors in the economy and therefore MAKRO is able to generate sluggish prices in both and also in the aggregate price level.

Table 4.1: Pricing and Markup Code Names

$P^0_{s,t}$	= pKLBR[s,t]	$\psi_{s,t}$	= upYTraeghed[sp]
$P_{s,t}$	= pY[sp,t]	$Y_{s,t}$	= qY[sp,t]
$\sigma_{s,t}$			
$\theta_{s,t}$	= srMarkup[sp,t]		

5 Labor market

The model of the labor market contains heterogeneous households and firms. Households choose the supply of hours and labor market participation. Labor demand comes from firms posting vacancies optimally. A matching technology brings vacancies and workers together. The model closes with bargaining between agents representing workers and firms which sets the market wage. The goal of the model is to reproduce the level and behavior of employment and wages.

There is a life cycle with workers of different ages on the household side, and there is sectoral disaggregation of production on the firm side. This is a large problem and in order to limit its size we build the model so that the household side is age specific, the firm side is sector specific, but the two dimensions are never present simultaneously. In addition, households have two types, the financially constrained and the unconstrained, and we solve the model so that both types have the same labor market decisions. The following are the key assumptions we make:

- Firms cannot choose who they hire. Firms and workers meet at random in the matching process, and once the meeting takes place it is never optimal to send the worker away, irrespective of how old the worker might be.

- Optimal labor market participation and hours are age specific but not sector specific. The worker cannot choose which sector she will be employed in, and cannot quit voluntarily a job in one sector to join a different one.

- For computational reasons we do not solve for the age distribution of workers inside each firm, and instead impose that it is always the same for every firm. The data shows that the average age of workers is the same across firms of different sizes, and also for firms that are expanding or shrinking in terms of labor force size.

5.1 Households

The timing convention is that all decisions are taken and production occurs at the end of each period. There is an exogenous number $N_{a,t}$ of households of age a in period t. In this chapter we interpret each household as containing of a unit mass of identical agents who share the risk of unemployment. Households aged a at time t survive into the following age and period with probability $s_{a,t}$. When a household dies its entire unit mass of members dies with it.

At the start of each period, agents in a household of age a can be in one of two labor market states: employed, $(1 - \delta_{a-1,t-1})\, q^e_{a-1,t-1}$, or not, $1-(1 - \delta_{a-1,t-1})\, q^e_{a-1,t-1}$, where $\delta_{a,t}$ is an exogenous age-specific job destruction rate. Given the two-state approach to the labor market there is no explicit notion of labor force in the model. Everyone is the labor market and the labor force is an exogenous construction we can use of´n the data generated by the model. After decisions are taken in period t and the market clears, a fraction $0 < q^e_{a,t} < 1$ of the household will be employed in period t.

Total employment, n_t , contains the employment of optimizing agents which are the residents in the country, $n^e_t = \sum_a q^e_{a,t} N_{a,t}$, plus an exogenous measure of migrant workers, n^f_t.

5.1.1 Utility function and budget constraint

Utility function The utility function from the optimal consumption decision, which is a function of consumption and housing, is now extended to include participation/search

$q^s_{a,t}$, and hours worked $h^e_{a,t}$.

$$U_{a,t} = U\left(C_{a,t}, D_{a,t}\right) - \left[\underbrace{Z^S_{a,t}}_{\text{Control}} \underbrace{\rho^e_{a,t}}_{\text{Scaling}} \underbrace{\lambda^n_{a,t} \frac{\left(q^s_{a,t}\right)^{1+\eta^n}}{1+\eta^n}}_{\text{Disutility from search}} + \underbrace{Z^H_{a,t}}_{\text{Control}} \underbrace{\rho^e_{a,t} q^e_{a,t}}_{\text{Scaling}} \underbrace{\lambda^h_{a,t} \frac{\left(h^e_{a,t}\right)^{1+\eta^h}}{1+\eta^h}}_{\text{Disutility from hours}} \right]$$

The terms $Z_{a,t}$ are utility weights taken as given by the household and used to control for stationarity, and to eliminate the marginal utility of consumption from the first order conditions.[72] This allows us to have the same search and hours decisions for constrained and unconstrained households, which greatly reduces model complexity.[73] The presence of constrained households is aimed primarily at the marginal propensity to consume out of an income shock, and is not thought of as having a significant impact on search decisions in the labor market.

The object $\rho^e_{a,t}$ is an individual worker productivity factor. $\lambda^n_{a,t}$ and $\lambda^h_{a,t}$ are disutility parameters.

Budget constraint We can think of participation in the labor market as a commitment to search for a job when not working, and of non-participation as the decision not to search - and therefore of not finding a job with probability one. We consider then that all agents in all households are in the "labor market" implying everyone searches for a job with some intensity. The search object $q^s_{a,t}$ can therefore be understood either as the number (fraction of the unit mass in the household) of workers searching for a job, or as a combined measure of number of agents searching times unobservable search intensity. In both cases this measure has a lower bound at zero and an upper bound $1 - \left(1 - \delta_{a-1,t-1}\right) q^e_{a-1,t-1}$.

The household takes as given the probability of getting a job, $\hat{x}_{a,t}$. By definition this is also the number of jobs obtained out of total labor market participation or total search effort. Therefore the law of motion for household employment is

$$q^e_{a,t} = \left(1 - \delta_{a-1,t-1}\right) q^e_{a-1,t-1} + \hat{x}_{a,t} q^s_{a,t} \tag{5.1}$$

Household searchers that find a job earn compensation $\tilde{w}_{a,t}$, the same as earned by those working who have kept their jobs from the previous period. This compensation is the after tax wage income for total hours worked:

$$\tilde{w}_{a,t} = \left(1 - \tau_{a,t}\right) \bar{w}_t \rho^e_{a,t} h^e_{a,t}$$

These are wages received by the worker. Below we define different wage objects for the firm and for the bargaining problem.

Those searching that fail to find a job earn compensation $b_{a,t} = r^b_{a,t} \tilde{w}_{a,t}$, where $r^b_{a,t}$ is a replacement ratio **function** (not just an exogenous proportion). The same compensation is earned by those that are neither working nor looking for a job, $1 - \left(1 - \delta_{a-1,t-1}\right) q^e_{a-1,t-1} - q^s_{a,t}$. The budget constraint is

$$p^c_t C_{a,t} = \hat{x}_{a,t} \tilde{w}_{a,t} q^s_{a,t} + \tilde{w}_{a,t} \left(1 - \delta_{a-1,t-1}\right) q^e_{a-1,t-1} + \left(1 - \hat{x}_{a,t}\right) b_{a,t} q^s_{a,t}$$
$$+ b_{a,t} \left(1 - \left(1 - \delta_{a-1,t-1}\right) q^e_{a-1,t-1} - q^s_{a,t}\right) + \Pi_{a,t}$$

and the term $\Pi_{a,t}$ summarizes all other objects. Collecting terms and using the law of motion to eliminate $q^s_{a,t}$ this expression becomes

$$p^c_t C_{a,t} = \left(\tilde{w}_{a,t} - b_{a,t}\right) q^e_{a,t} + b_{a,t} + \Pi_{a,t}$$

[72] Galí, Smets, and Wouters (2012).

[73] Preserving the wealth effect for constrained households and eliminating the labor supply from unconstrained households would yield a model of entrepreneurs and workers as in Pedersen (2016).

which says that every household member earns b, and working members earn an additional wage premium over b. It is this wage premium that determines the incentive to search for a job and participate in the labor market. The unemployment compensation is in itself irrelevant and only matters to the extent that it changes the wage premium. If wages responded one to one to changes in b there would be no change in the search effort.

There is a subtle point which deserves further clarification. When deriving and simplifying the budget constraint it is useful to consider the variable $q^s_{a,t}$ as measured in numbers of household members looking for a job. However, if understood as a total search effort object, the variable $q^s_{a,t}$ is not measured in the same unit as $q^e_{a,t}$ even though it is bounded between zero and $1-(1-\delta_{a-1,t-1})\,q^e_{a-1,t-1}$. In this case it is only the product $\hat{x}_{a,t}q^s_{a,t}$ which is measured in the same units as $q^e_{a,t}$, namely number of workers. This product is still well behaved because below we model the job finding rate to lie in the unit interval, so no boundaries are ever crossed.

We are now ready to maximize utility subject to the budget constraint and obtain the first order conditions for participation and hours.

5.1.2 Optimal choice of hours

Hours vary by age in the data and so the disutility parameter $\lambda^h_{a,t}$ varies with age to calibrate this pattern. The first order condition is

$$\frac{\partial U_{a,t}}{\partial C_{a,t}}\left[1-\tau_{a,t}\right]\frac{\bar{w}_t}{p^c_t} = \underbrace{Z^{ch}_{a,t}Z^{wh}_{a,t}}_{Z^H_{a,t}}\lambda^h_{a,t}\left(h^e_{a,t}\right)^{\eta^h}$$

The term $Z^{wh}_{a,t}$ is used to control for trends in after-tax real wages, such that the first order condition is stationary and does not drift towards a corner solution. The term Z^{ch}_t is used to eliminate the wealth effect from this equation.[74] This implies we have the same optimality condition for financially constrained and unconstrained households. We consider real wage short run deviations from the long run path,

$$Z^{wh}_{a,t} = \lambda^{zwh}Z^{wh}_{a-1,t-1} + \left(1-\lambda^{zwh}\right)\left(1-\tau_{a,t}\right)\frac{\bar{w}_t}{p^c_t}$$

but we do not consider short term deviations in the marginal utility of consumption

$$Z^{ch}_t = \frac{\partial U_{a,t}}{\partial C_{a,t}}$$

We have then

$$\frac{1}{Z^{wh}_{a,t}}\left(1-\tau_{a,t}\right)\frac{\bar{w}_t}{p^c_t} = \lambda^h_{a,t}\left(h^e_{a,t}\right)^{\eta^h}$$

In the long run the ratio of wages to the Z term disappears from this first order condition:

$$1 = \lambda^h_{a,t}\left(h^e_{a,t}\right)^{\eta^h}$$

where hours respond only to preferences and do so with a very low elasticity since $\eta^h = 11$.[75]

[74] The standard is to assume $Z^c_{a,t}$ is a function of average marginal utility which in symmetric equilibrium equals the individual marginal utility. With the pooling assumptions within the unit mass of each household, and the assumption that all households are identical the average and the marginal are always identical but the symmetric equilibrium concept remains.

[75] We do not model the long run downward trend of the workweek. Z^{ch}_t rules out the income effect (higher consumption implying lower marginal utility) and the long run Z^{wh}_t rules out the substitution effect of higher taxes (funding the expanding welfare state).

5.1.3 Optimal choice of search

As the job finding rate is exogenous to the household we can solve the problem by choosing directly $q^e_{a,t}$. The first order condition for $q^e_{a,t}$ is:

$$\frac{\partial U_{a,t}}{\partial C_{a,t}} \frac{(\tilde{w}_{a,t} - b_{a,t})}{\tilde{w}_{a,t}} \frac{\tilde{w}_{a,t}}{p^c_t} = Z^H_{a,t}\rho^e_{a,t}\lambda^h_{a,t}\frac{\left(h^e_{a,t}\right)^{1+\eta^h}}{1+\eta^h} + Z^S_{a,t}\Gamma_{a,t} - \beta_{a,t}\left(1-\delta_{a,t}\right)s_{a,t}Z^S_{a+1,t+1}\Gamma_{a+1,t+1} \tag{5.2}$$

where

$$\Gamma_{a,t} = \frac{\lambda^n_{a,t}\rho^e_{a,t}\left[q^s_{a,t}\right]^{\eta^n}}{\hat{x}_{a,t}}$$

where the survival rate $s_{a,t}$ factors the term in $t+1$, and where $\beta_{a,t}$ is the utility discount factor. Optimality trades off current against future marginal utility. Extra engagement in the labor market today will result in additional employment with associated payoff $\tilde{w} - b$. There is an immediate downside from the additional disutility of hours and of participation, but there is also a savings term from the fact that, next period, $(1-\delta)$ of the additional employment found today will remain at work implying no disutility from looking for a job.

5.1.4 Short and Long Run Algebra

Define here $Z^S_{a,t} = Z^{cs}_{a,t}Z^{ws}_{a,t}$ and assume the same consumption factor as in the hours term, $Z^{cs}_{a,t} = Z^{ch}_{a,t}$ (the cohort average of the marginal utility of consumption). Divide through by $Z^S_{a,t}$ and use the hours foc to get

$$\left[1 - r^b_{a,t} - \frac{1}{1+\eta^h}\right]\rho^e_{a,t}\left[\frac{1}{Z^{ws}_{a,t}}\left(1-\tau_{a,t}\right)\frac{\bar{w}_t}{p^c_t}h^e_{a,t}\right] = \Gamma_{a,t} - \beta_{a,t}\left(1-\delta_{a,t}\right)s_{a,t}\frac{Z^S_{a+1,t+1}}{Z^S_{a,t}}\Gamma_{a+1,t+1}$$

The short run wage factor in this equation is not identical to the wage factor in the hours equation:

$$Z^{ws}_{a,t} = \lambda^{zws}Z^{wh}_{a-1,t-1} + \left(1-\lambda^{zws}\right)\left(1-\tau_{a,t}\right)\frac{\bar{w}_t}{p^c_t}h^e_{a,t}\left[1 - r^b_{a,t} - \frac{1}{1+\eta^h}\right]^{1-\eta^n/\eta^b}$$

In particular, we extend the definition of the $Z^{ws}_{a,t}$ term to include $\left[1 - r^b_{a,t} - \frac{1}{1+\eta^h}\right]^{1-\eta^n/\eta^b}$ which allows us to match the estimated long run elasticity of the labor supply with respect to changes in the benefit ratio r^b_a, independently of the short run movements in labor supply.

The definition of the short run $Z^{cs}_{a,t}$ terms is now important. We eliminate the marginal utility of consumption from the hours decison because it is a static decision, but here it resurfaces on the right hand side through

$$\frac{Z^{cs}_{a+1,t+1}}{Z^{cs}_{a,t}} = \frac{\frac{\partial U_{a+1,t+1}}{\partial C_{a+1,t+1}}}{\frac{\partial U_{a,t}}{\partial C_{a,t}}}$$

This not only brings back the wealth effect that we are eliminating, it also implies HTM and forward looking households have different search decisions, which is an additional heterogeneity we do not want to include in the model. We therefore approximate the factor $Z^{cs}_{a+1,t+1}/Z^{cs}_{a,t}$ with the average of this quantity for forward looking agents in the calibration years. And assume that marginal utility of consumption behaves identically for HTM and forward looking agents.[76] This factor is then fixed and is identical in the short and long run.

Replacing for $\Gamma_{a,t}$ and dividing by $\rho^e_{a,t}$ the f.o.c. in the long run is given by

$$\left[1 - r^b_a - \frac{1}{1+\eta^h}\right]^{\eta^n/\eta^b} = [q^s_a]^{\eta^n} \cdot \underbrace{\frac{\lambda^n_a}{\hat{x}_a}\left(1 - (1-\delta_a)\, s_a \beta_{a,t} \frac{Z^S_{a+1}}{Z^S_a} \frac{\rho^e_{a+1}}{\rho^e_a} \frac{\frac{\lambda^n_{a+1}}{\lambda^n_a}\left[\frac{q^s_{a+1}}{q^s_a}\right]^{\eta^n}}{\frac{\hat{x}_{a+1}}{\hat{x}_a}}\right)}_{\Phi_a}$$

This expression allows us to obtain a back of the envelope measure of the elasticity of the labor supply with respect to the replacement ratio. Taking all objects inside Φ_a as exogenous we take logs and differentiate to obtain

$$\frac{1}{\eta^b} \cdot dlog\left(1 - r^b_a - \frac{1}{1+\eta^h}\right) \approx dlog\left(q^s_a\right)$$

Finally, we currently implement the long run version of this equation even in the main model, in the sense that the real wage does not explicitly affect search effort in this first order condition.

5.1.5 Labor supply elasticities

Given our assumptions we obtain a static (short term) elasticity of hours with respect to wages, $\frac{1}{\eta^h}$. This is, however, not the elasticity of total labor supply. Define $n^e_t = q^e_t N_t$. The object of interest is

$$\frac{dlog\left(n^e_t h^e_t\right)}{dlog\left(\frac{w_t}{p^c_t}\right)} = \frac{dlog\left(n^e_t\right)}{dlog\left(\frac{w_t}{p^c_t}\right)} + \frac{1}{\eta^h}$$

Employment is an indirect consequence of participation. Also, the job finding rate changes in equilibrium following an exogenous shock. As discussed in Attanasio et al. (2018) it is only possible to map structural parameters to labor supply responses to shocks by running the entire model.

5.1.6 Aggregation

Population flows obey

$$N_{a,t} = N_{a-1,t-1} s_{a-1,t-1} + I_{a,t} - E_{a,t}$$

where $I_{a,t}$ are immigrants and $E_{a,t}$ are emigrants. Households making the choice described above are those surviving from the previous period. Not just that, they are the ones surviving which stay in the country, $N_{a-1,t-1} s_{a-1,t-1} - E_{a,t}$. Emigrants $E_{a,t}$ are just like residents, except they leave. With this in mind we have for these remaining agents

$$q^e_{a,t} = \left(1 - \delta_{a-1,t-1}\right) q^e_{a-1,t-1} + \hat{x}_{a,t} q^s_{a,t}$$

Immigrants $I_{a,t}$ come into the country and we assume they obtain the same employment $q^e_{a,t}$ as residents. However, they do not have an employment history in the country. Furthermore we assume some immigrants come already with a job so they do not have to search. This accounts for the employment quantity $q^I_{a,t} I_{a,t}$. We have then

$$q^e_{a,t} = \hat{x}_{a,t} q^s_{a,t} + q^I_{a,t}$$

[76] We model HTM households in reduced form so that we do not explicitly specifiy their utility.

This sums to

$$\underbrace{N_{a,t}q^e_{a,t}}_{n^e_{a,t}} = (1-\delta_{a-1,t-1})\left(s_{a-1,t-1} - \frac{E_{a,t}}{N_{a-1,t-1}}\right) q^e_{a-1,t-1} N_{a-1,t-1}$$
$$+ \hat{x}_{a,t} \underbrace{q^s_{a,t}\left(s_{a-1,t-1}N_{a-1,t-1} - E_{a,t} + I_{a,t}\right)}_{n^s_{a,t} \equiv N_{a,t}q^s_{a,t}} + q^I_{a,t} I_{a,t}$$

Now assume that

$$q^I_{a,t} \equiv (1-\delta_{a-1,t-1})\, q^e_{a-1,t-1}$$

so that

$$n^e_{a,t} = (1-\delta_{a-1,t-1})\left(\underbrace{s_{a-1,t-1} - \frac{E_{a,t}}{N_{a-1,t-1}} + \frac{I_{a,t}}{N_{a-1,t-1}}}_{\frac{N_{a,t}}{N_{a-1,t-1}}}\right) n^e_{a-1,t-1} + \hat{x}_{a,t} n^s_{a,t}$$

which allows us to define the cohort aggregate destruction rate as $\hat{\delta}_{a,t}$ such that

$$n^e_{a,t} = \underbrace{(1-\delta_{a-1,t-1})\frac{N_{a,t}}{N_{a-1,t-1}}}_{1-\hat{\delta}_{a,t}} n^e_{a-1,t-1} + \hat{x}_{a,t} n^s_{a,t} = \left(1-\hat{\delta}_{a,t}\right) n^e_{a-1,t-1} + \hat{x}_{a,t} n^s_{a,t}$$

With this construction we do not have to know the number of immigrants and emigrants. All we need to know is total population. We have then two objects: total search effort $n^s_{a,t} \equiv N_{a,t}q^s_{a,t}$ and the redefined population job destruction rate $\hat{\delta}_{a,t}$. Note that this is different from the job destruction rate that matters for the individual optimization problem, $\delta_{a,t}$.

This construction links with the destruction rate which is relevant for the firm. With the additional assumptions we make in the problem of the firm, the only job destruction rate that matters is the one aggregated over the age distribution, δ^n_t, and which is identical for all firms:

$$(1-\delta^n_t) = \frac{\sum_a \left(1-\hat{\delta}_{a,t}\right) n^e_{a-1,t-1}}{\sum_a n^e_{a-1,t-1}} = \frac{\sum_a \left(1-\hat{\delta}_{a,t}\right) n^e_{a-1,t-1}}{n^e_{t-1}}$$

so that

$$n^e_t = (1-\delta^n_t)\, n^e_{t-1} + \hat{x}_t n^s_t$$

It is useful at this point to collect some of the large number of objects in the model and describe them as they are present in the code. This is contained in Table 1.

Table 5.1: Labor market code names: Households

η^n	eDeltag
η^h	eh
δ_a	rSeparation[a,t]
δ_t^n	rSeparation[aTot,t]
$\hat{x}_{a,t}$	rJobFinding[t]
$n_{a,t}^e$	nLHh[a,t]
$Z_{a,t}^{wh}$	fZh[a,t]
$\lambda_{a,t}^n$	uDeltag[a,t]
$\lambda_{a,t}^h$	uh[a,t]
$\rho_{a,t}^e$	fProdHh[a,t]
$\tau_{a,t}$	mtInd[a,t]
$h_{a,t}^e$	hLHh[a,t]
$r_{a,t}^b$	mrKomp[a,t]
$n_{a,t}^s$	nSoegHh[a,t]

5.1.7 Migrant workers

Total employment includes both employed who are residents in Denmark, and migrant workers who are not. The households whose decisions we have detailed are resident households. However, firms in the model do not distinguish between residents and migrants when they hire. Migrant workers are cross border agents that work in Denmark, but live abroad most or all of the year. These migrant workers are not the immigrants $I_{a,t}$ described above as those are part of the resident population $N_{a,t}$ and also consume and save. Migrant workers provide search input $n_{a,t}^{s,f}$ into the matching function and generate employment $n_{a,t}^{f}$. They face the same job destruction rates and die and migrate at the same rate as the locals and they stay in their jobs when these are not destroyed, but do not demand local consumption or housing. However, they may have different productivity and work different hours from the locals.

Migrant workers have the same probability of finding a job as local job searchers, $\hat{x}_t$, and the number of employed migrant workers obeys the law of motion

$$n_t^f = \left(1 - \hat{\delta}_t\right) n_{t-1}^f + \hat{x}_t n_t^{s,f}$$

The total number of cross border persons who are either employed or searching for a job in Denmark can be written as

$$N_t^f = \left(1 - \hat{\delta}_t\right) n_{t-1}^f + n_t^{s,f}$$

We assume that this total is exogenous. The number of migrant workers searching for a job is then endogenous and given by

$$n_t^{s,f} = N_t^f - \left(1 - \hat{\delta}_t\right) n_{t-1}^f$$

E.g. when the job finding rate is higher, more of the potential migrant workers find employment, reducing the search input of migrant workers in the following period.

Since migrant workers only enter the model through the firm and the matching function, their age decomposition is irrelevant and only their aggregate contribution matters, but accounting for the age variation makes the algebra below more transparent. We assume then that their age distribution is identical to that of residents.

Table 5.2: Labor market code names: Migrant workers

n_t^f	nLUdl[t]
$n_t^{s,f}$	nSoegUdl[t]
N_t^f	nSoegBaseUdl[t]

5.2 Firms

What follows applies to all private sectors in the model. The public sector is treated differently. There is a unit mass of identical firms in each (private) sector j. Sectors are indexed by the letter s in the code, but as we use s here to denote search, we keep with the general index letters i and j throughout. Employment in the firm is given by the measure n_t which sums residents and migrant workers.

Total workers in sector j, $n_{j,t}$, contribute with a total amount of productive hours given by $\bar{\rho}_t \bar{h}_t n_{j,t}$, where $\bar{\rho}$ is the productivity factor and $\bar{h}$ is the hours factor in the firm. The bar in $\bar{\rho}$ and $\bar{h}$ distinguishes the object inside the firm from the one obtained in the household optimization above.

Firms post vacancies and the economy wide matching function m_t dictates their success in filling them. This process occurs in period t, and, after it is completed, employment for the current period is determined and production occurs at the end of the period. The firm cannot affect the hours worked by its employees and takes them as given. An effort/utilization choice by the firm is added to the model to help generate procyclical value added per worker but that choice is detailed in the chapter on firms.

5.2.1 Objects

Wages paid by firms $\hat{w}$ contain payroll taxes which are adjusted for the fraction of self employed $\tau_t^L(1-r_{j,t}^{self})$, the actual wage paid $\bar{w}_t$, the sectoral relative wage factor $\rho_{j,t}^w$, as well as the average productivity and hours aggregates $\bar{\rho}_t, \bar{h}_t$.

$$\hat{w}_{j,t} = \bar{w}_t \left(1 + \tau_t^L \left(1 - r_{j,t}^{self}\right)\right) \rho_{j,t}^w \bar{\rho}_t \bar{h}_t$$

Another useful object is the total amount of productive labor input into production, L. This is the object inside the production function. It contains an exogenous labor augmenting productivity factor z_t, the endogenous utilization factor u_t, the sectoral relative wage factor $\rho_{j,t}^w$, the individual productivity and individual hours aggregates $\bar{\rho}_t \bar{h}_t$, and finally contains the endogenous correction for the fraction of employment used in the hiring process which we denote by χ:

$$L_{j,t} = z_{j,t} u_{j,t} \rho_{j,t}^w \bar{\rho}_t \bar{h}_t \left(1 - \chi_{j,t}\right) n_{j,t}$$

The cost of hiring is defined in terms of units of labor lost to production so that the total number of heads actually producing output is given by $(1-\chi)n$. For the algebra below we collect several terms in one auxiliary object ξ:

$$\xi_{j,t} = z_{j,t} u_{j,t} \rho_{j,t}^w \bar{\rho}_t \bar{h}_t$$

Finally, the choice variable for the firm is the number of workers, so that the relevant derivatives are

$$\frac{\partial L_{j,t}}{\partial n_{j,t}} = \xi_{j,t} \left(1 - \frac{\partial \left(\chi_{j,t} n_{j,t}\right)}{\partial n_{j,t}}\right)$$

$$\frac{\partial L_{j,t+1}}{\partial n_t^j} = -\xi_{j,t+1} \frac{\partial \left(\chi_{j,t+1} n_{j,t+1}\right)}{\partial n_{j,t}}$$

and we now discuss in more detail the object χ.

5.2.2 Vacancy posting costs

A unit mass of firms in sector j posts vacancies v and the law of motion for employment n is[77]

$$n_{j,t} = (1 - \delta_t^n)\, n_{j,t-1} + m_t v_{j,t}$$

The cost of posting vacancies is incurred in units of employment. We define an auxiliary endogenous variable χ such that $\chi_{j,t} n_{j,t}$ equals total vacancy posting costs, which contain a linear and a quadratic component:

$$\chi_{j,t} n_{j,t} = \kappa v_{j,t} + \lambda m_t v_{j,t} + \frac{\gamma}{2} n_{j,t} \left[\frac{n_{j,t}}{n_{j,t-1}} / \frac{n_{j,t-1}}{\overline{n}_{j,t-2}} - \alpha_t \right]^2$$

$$\frac{\partial \left(\chi_{j,t} n_{j,t}\right)}{\partial n_{j,t}} = \frac{\kappa}{m_t} + \lambda \frac{m_t v_{j,t}}{n_{j,t-1}} + \underbrace{\frac{\gamma}{2} \left[\frac{n_{j,t}}{n_{j,t-1}} / \frac{n_{j,t-1}}{\overline{n}_{j,t-2}} - \alpha_t \right]^2}_{\approx 0} + \gamma \frac{n_{j,t}}{n_{j,t-1}} / \frac{n_{j,t-1}}{\overline{n}_{j,t-2}} \left[\frac{n_{j,t}}{n_{j,t-1}} / \frac{n_{j,t-1}}{\overline{n}_{j,t-2}} - \alpha_t \right]$$

$$\frac{\partial \left(\chi_{j,t} n_{j,t}\right)}{\partial n_{j,t-1}} = -\left(1 - \delta_t^n\right) \left[\frac{\kappa}{m_t} + \lambda \right] - 2\gamma \frac{n_{j,t}}{n_{j,t-1}} \left[\frac{n_{j,t}}{n_{j,t-1}} / \frac{n_{j,t-1}}{\overline{n}_{j,t-2}} \right] \left[\frac{n_{j,t}}{n_{j,t-1}} / \frac{n_{j,t-1}}{\overline{n}_{j,t-2}} - \alpha_t \right]$$

and we note that the quadratic term is always approximately zero although its derivatives matter, as despite being small they are linear objects.

5.2.3 Choosing employment. The user cost of labor.

When firms post vacancies they hire workers. Although workers are "attached" to hours and productivity, the choice variable for the firm is n as the firm takes m as given. Current profits (as relevant for the optimal employment decision) are

$$\pi_t^j = \left(1 - \tau_{j,t}^c\right) \left\{ p_{j,t}^0 Q^j \left(L_{j,t}\right) - \hat{w}_{j,t} n_{j,t} \right\}$$

where corporate taxes are explicit but other, possibly sector specific, taxes and subsidies are implicit in wages paid, and in prices p_t^j.

The first order condition for employment is

$$\left(1 - \tau_{j,t}^c\right) p_{j,t}^L \frac{\partial L_{j,t}}{\partial n_{j,t}} + \left(1 - \tau_{j,t+1}^c\right) \beta_{t+1} p_{j,t+1}^L \frac{\partial L_{j,t+1}}{\partial n_{j,t}} - \left(1 - \tau_{j,t}^c\right) \hat{w}_{j,t} = 0 \tag{5.3}$$

where we define a user cost variable to be the value of the marginal physical product, evaluated at the optimization price p^0:[78]

$$p_{j,t}^L \equiv p_{j,t}^0 Q_{L_t}^j$$

This is the user cost of the object L. But L is not "labor". It is here the total utilization made of the effective labor input in productivity units. Furthermore, as our model of posting vacancies accounts for these costs inside the L object, the intuition behind the user cost becomes less transparent. Nevertheless the user cost is, as usual, the wage plus a positive term which reflects the costs of hiring, so that it is approximately $p^L \approx (1+\chi)w$. Finally, the average value of χ is linked to the value of the bargaining power parameter in the wage setting part of the model.

[77]Given an identical age distribution inside all firms, the objects $\left(\delta^n, \bar{\rho}, \bar{h}\right)$ are not sector specific. Nominal wages w and the matching rate m are also aggregate objects.

[78]This optimization price is derived in the production chapter.

5.2.4 Algebra

It is useful to make some of these terms explicit as it will make the first order condition resemble the code. Detailing the L derivatives and dividing by ξ we obtain

$$p^L_{j,t}\left(1-\frac{\partial\left(\chi_{j,t}n_{j,t}\right)}{\partial n_{j,t}}\right)=\frac{\hat{w}_{j,t}}{\xi_{j,t}}+D^n_{j,t+1}p^L_{j,t+1}\left(\frac{\partial\left(\chi_{j,t+1}n_{j,t+1}\right)}{\partial n_{j,t}}\right)$$

where $D^n_{j,t+1}$ is a discount factor

$$D^n_{j,t+1}\equiv\beta_{t+1}\frac{\left(1-\tau^c_{j,t+1}\right)}{\left(1-\tau^c_{j,t}\right)}\frac{\xi_{j,t+1}}{\xi_{j,t}}=\beta_{t+1}\frac{\left(1-\tau^c_{j,t+1}\right)}{\left(1-\tau^c_{j,t}\right)}\frac{z_{j,t+1}u_{j,t+1}\rho^w_{j,t+1}\bar{\rho}_{t+1}\bar{h}_{t+1}}{z_{j,t}u_{j,t}\rho^w_{j,t}\bar{\rho}_t\bar{h}_t}$$

and since a number of terms cancel each other in the $\hat{w}_{j,t}/\xi_{j,t}$ ratio we obtain

$$\frac{\hat{w}_{j,t}}{\xi_{l,t}}=\bar{w}_t\frac{\left(1+\tau^L_t\left(1-r^{self}_{j,t}\right)\right)}{z_{j,t}u_{j,t}}=\bar{w}_t\Gamma^L_{j,t}$$

The object $\hat{w}_{j,t}/\xi_{j,t}$ has the same growth properties as the left hand side object $p^L_{j,t}$. The user cost object $p^L_{j,t}$ is a price and grows at the rate π which we use to correct price growth in the entire model. The variable $z_{j,t}$ contained inside $\xi_{j,t}$ grows at the real rate g. This implies the wage $\bar{w}_t$ is not just a price but rather a “value” object which must grow at the rate $(1+g)(1+\pi)$. This also explains why in the code this wage $\bar{w}_t$ is denominated ”vw” rather than just ”w”.[79]

A final note regarding implementation is in order. Even though hiring is a dynamic forward looking decision, in the response to shocks the first order condition of the firm is transformed into a quasi static expression by exogenizing the object

$$\beta_{t+1}\frac{p^L_{j,t+1}}{p^L_{j,t}}$$

which appears inside the composite

$$D^n_{j,t+1}\frac{p^L_{j,t+1}}{p^L_{j,t}}$$

as it enters the first order condition

$$1-\frac{\partial\left(\chi_{j,t}n_{j,t}\right)}{\partial n_{j,t}}=\frac{\hat{w}_{j,t}}{p^L_{j,t}\xi_{j,t}}+\underbrace{D^n_{j,t+1}\frac{p^L_{j,t+1}}{p^L_{j,t}}}\left(\frac{\partial\left(\chi_{j,t+1}n_{j,t+1}\right)}{\partial n_{j,t}}\right)$$

such that this part of the discount factor remains as it is in the baseline when we shock the model. The first order condition retains a modicum of forward looking behavior since the derivative $\partial\left(\chi_{j,t+1}n_{j,t+1}\right)/\partial n_{j,t}$ contains endogenous variables.

5.2.5 Link to CES optimization

The dynamic first order condition provides the input to the CES minimization problem used in solving the overall problem of the firm. The CES function is

$$p^{kl,j}_tQ^{kl}_{j,t}\equiv p^{kl,j}_t\left[\left(\mu^k_{j,t}\right)^{\frac{1}{E}}\left(u^k_{j,t}K_{j,t}\right)^{\frac{E-1}{E}}+\left(\mu^l_{j,t}\right)^{\frac{1}{E}}\left(L_{j,t}\right)^{\frac{E-1}{E}}\right]^{\frac{E}{E-1}}$$

[79]In order to correct for growth in the code, on the right hand side of the first order condition the future user cost is defined exactly as here and mutiplied by a factor $1+\pi$ as in the following example: $p^L_t=w_t+\beta(1+\pi)p^L_{t+1}$.

On the budget side of the CES problem we have the total cost associated with all the labor actually used, l:

$$p^{kl}_{j,t} Q^{kl}_{j,t} \equiv p^{L}_{j,t} L_{j,t} + p^{K}_{j,t} K_{j,t}$$

where the object $p^{L}_{j,t}$ is the user cost of L. In the CES optimization problem we take a derivative with respect to $L_{j,t}$ and this yields

$$L_{j,t} = \mu^{l}_{j,t} Q^{kl}_{j,t} \left(\frac{p^{L}_{j,t}}{p^{kl}_{j,t}} \right)^{-E}$$

The last identity ensures the CES problem is consistent with the optimization problem and shows the relationship between the user cost of L and the optimal choice of vacancies.

The companion expression for capital is of course

$$u^{k}_{t} K_{j,t} = \mu^{k}_{j,t} Q^{kl}_{j,t} \left(\frac{p^{K}_{j,t}}{p^{kl}_{j,t}} \frac{1}{u^{k}_{t}} \right)^{-E}$$

Table 5.3: Labor market code names: Firms

$n_{s,t}$	nL[s,t]
κ_t	uOpslagOmk
γ	uOpslagOmkSqr
m_t	rMatch[t]
$\chi_{s,t}$	rOpslagOmk[s,t]
$\frac{\partial(\chi_{s,t} n_{s,t})}{\partial n_{s,t}}$	dOpslagOmk2dnL[s,t]
L_t	qL[t]
$\frac{\partial L_{s,t+1}}{\partial n_{s,t}}$	dqLLead2dnL[s,t]
$p^L_{s,t}$	pL[s,t]
$t^c_{s,t}$	tSelskab[t]
$\frac{1}{\beta_t} - 1$	rVirkDisk[t]
$u_{s,t}$	rLUdn[s,t]
η_u	eLUdn

As we did in the household part, we now collect some of the objects from this section and equate them to their descriptions in the code. This is contained in Table 4.

Table 5.4: Labor market code names: Firms

$n_{s,t}$	nL[s,t]
κ_t	uOpslagOmk
γ	uOpslagOmkSqr
m_t	rMatch[t]
$\chi_{s,t}$	rOpslagOmk[s,t]
$\frac{\partial(\chi_{s,t} n_{s,t})}{\partial n_{s,t}}$	dOpslagOmk2dnL[s,t]
L_t	qL[t]
$\frac{\partial L_{s,t+1}}{\partial n_{s,t}}$	dqLLead2dnL[s,t]
$p^L_{s,t}$	pL[s,t]
$t^c_{s,t}$	tSelskab[t]
$\frac{1}{\beta_t} - 1$	rVirkDisk[t]
$u_{s,t}$	rLUdn[s,t]
η_u	eLUdn

5.3 The matching friction

Rather than modelling the rate of filling a vacancy with a worker of a given age, $\hat{m}_{a,t}$, we model instead the probability of finding a job, $\hat{x}_{a,t}$ as

$$\hat{x}_{a,t} = \mu_a \left[\frac{\left(\frac{v_t}{n_t^{s,agg}}\right)^{\alpha}}{1 + \left(\frac{v_t}{n_t^{s,agg}}\right)^{\alpha}} \right] \tag{5.4}$$

Inside this function the total search effort of all workers looking for a job, $n_t^{s,agg}$, is matched against total vacancies created across all firms in the economy $v_t = \sum_j v_{j,t}$. The ratio variable is the aggregate market tightness variable often labelled θ_t.

We set $1 \geq \mu_a > 0$ so the job finding probability is bounded between 0 and 1.[80] It is then impossible to find a job for all available workers, since a job finding rate of 1 only obtains if the ratio $\frac{v_t}{n_t^{s,agg}}$ is infinite. We consider $1 \geq \alpha > 0$. Setting $\alpha < 1$ dampens the response of employment to shocks.

We now make the simplifying assumption that the job finding rate is identical for all ages.[81] We set the parameter $\mu_a = \mu = 1$ and therefore $\hat{x}_{a,t} = \hat{x}_t$. The object $\hat{m}_{a,t}$ can now be defined through the identity

$$n_{a,t}^{s,agg} \hat{x}_t = \hat{m}_{a,t} v_t \tag{5.5}$$

and aggregating over ages $\sum_a n_{a,t}^{s,agg} = n_t^{s,agg}$, so that

$$\hat{x}_t n_t^{s,agg} = \sum_a \hat{m}_{a,t} v_t = m_t v_t$$

Given total vacancies we can invert this relationship and write the aggregate rate of filling a vacancy m_t. Unlike the job finding rate, this quantity m_t is not bounded above by 1 and therefore cannot be called a probability. However, the model is calibrated such that it is less than 1.

5.4 Aggregation algebra

5.4.1 Quantities

For a given firm, employment of workers aged a is the sum of resident and migrant workers $n_{a,t} = n_{a,t}^e + n_{a,t}^f$, where $n_{a,t}^e = q_{a,t}^e N_{a,t}$. Migrant workers are allocated proportionately across sectors so that the following equations apply to all sectors.[82]

$$n_t = n_t^e + n_t^f = \sum_a n_{a,t}^e + n_t^f = \sum_a q_{a,t}^e N_{a,t} + n_t^f$$

and total search effort is

$$n_t^{s,agg} = \sum_a n_{a,t}^{s,agg} = \underbrace{\underbrace{n_t^{s,f}}_{\text{Migrants}} + \sum_a q_{a,t}^s N_{a,t}}_{\text{Residents}} = \underbrace{n_t^{s,f}}_{\text{Migrants}} + \underbrace{\sum_a n_{a,t}^s}_{n_t^s \text{ Residents}}$$

[80] Petrongolo and Pissarides (2001) provide a survey of the matching function.

[81] This is possible because the search effort variable is not strictly a measure of the number of workers looking for a job.

[82] Total employment varies by sector, but we force age distributions within firms to be the same for all firms in all sectors.

5.4.2 Averages

Hours, productivity, tax rates We make additional proportionality assumptions. Hours of migrant workers are uniformly different by the factor μ_h and productivity is uniformly different by the factor μ_ρ. We can compute average hours in the firm from the identity $\bar{h}_t n_t = \sum_a \left(h^e_{a,t} n^e_{a,t} + h^f_{a,t} n^f_{a,t} \right)$ as

$$\bar{h}^e_t = \frac{\sum_a \left(h^e_{a,t} n^e_{a,t} \right)}{n^e_t}$$

$$\bar{h}_t = \left[\frac{n^e_t + \mu_h n^f_t}{n_t} \right] \bar{h}^e_t$$

The average value of the product ρh for residents is given by

$$\bar{\rho}^e_t \bar{h}^e_t = \frac{\sum_a h^e_{a,t} \rho^e_{a,t} n^e_{a,t}}{\sum_a n^e_{a,t}} = \frac{\sum_a h^e_{a,t} \rho^e_{a,t} n^e_{a,t}}{n^e_t}$$

Foreign workers have a different average value of this object

$$\bar{\rho}^f_t \bar{h}^f_t = \frac{\sum_a \mu_h h^e_{a,t} \mu_\rho \rho^e_{a,t} n^f_{a,t}}{\sum_a n^f_{a,t}} = \mu_\rho \mu_h \bar{\rho}^e_t \bar{h}^e_t$$

where the identity depends on the assumption of identical age distributions for residents and foreigners. The overall average factor for the firm depends on the weight of the different populations:

$$\bar{\rho}_t \bar{h}_t = \frac{n^e_t \bar{\rho}^e_t \bar{h}^e_t + n^f_t \bar{\rho}^f_t \bar{h}^f_t}{n^e_t + n^f_t} = \frac{n^e_t + \mu_\rho \mu_h n^f_t}{n_t} \bar{\rho}^e_t \bar{h}^e_t$$

The average income tax can be defined through

$$\bar{\tau}_t = \frac{\sum_a \tau_{a,t} \rho^e_{a,t} h^e_{a,t} n^e_{a,t} + \sum_a \tau_{a,t} \rho^f_{a,t} h^f_{a,t} n^f_{a,t}}{\bar{\rho}_t \bar{h}_t n_t}$$

5.4.3 Law of motion

The firm has employment n_t and has a job destruction rate given by

$$n_t = (1 - \delta^n_t)\, n_{t-1} + m_t v_t$$

and since we make the necessary assumption to ensure migrants have the same law of motion as residents we can write

$$1 - \delta^n_t = \frac{\sum_a (1 - \delta_{a-1,t-1}) \frac{N_{a,t}}{N_{a-1,t-1}} n^e_{a-1,t-1}}{\sum_a n^e_{a-1,t-1}} = \frac{\sum_a \left(1 - \hat{\delta}_{a,t} \right) n^e_{a-1,t-1}}{n^e_{t-1}}$$

where the aggregate destruction rate δ^n_t is now endogenous (although exogenous to the firm). Because the firm cannot choose who it hires, it effectively always hires the average job searcher. Then, as we impose the same age distribution inside every firm irrespective of sector, all firms are the same in this respect and they all face the same job destruction rate. Now, since they do not control who they hire, they do not control the job destruction rate either.

5.5 Wage determination

We need one last object to close the model. We have derived the equations determining optimal search/participation (the "size of the market") and optimal vacancy posting (labor demand).[83] The current macroeconomics benchmark is to close the model with search and bargaining. We use the Nash bargaining model.

We assume a unique bargaining agent on either side of the market which aggregates, on one side, preferences of all firms from all sectors, and, on the other, preferences of all workers of all ages. We assume also that these agents, which we can call unions, are "distant" from their individual firm and worker constituents so that they can solve a simplified problem on their behalf. These assumptions allow for a degree of freedom in setting up the surpluses that enter the bargaining problem.

5.5.1 Wage rigidity

The Nash solution yields proportionality between wages and productivity. A static example illustrates this point. Consider the firm surplus to be $J = y - w$, and the worker surplus $W = w - b = w\left(1 - r^b\right)$ where the unemployment benefit is proportional to the wage. The Nash solution then yields a constant ratio $\frac{w}{y}$. There is no wage rigidity w.r.t. changes in y.

Short run nominal wage rigidity is then added via a mechanism from Galí and Gertler (1999), where a fraction $(1-\gamma)(1-\theta^w)$ of contracts is renegotiated via bargaining with associated wage ω, and a second fraction of contracts $(1-\gamma)\theta^w$ adjusts in a mechanical way.[84] The relevant wage for firms and households is now an average wage, $\bar{w}_t$ which follows

$$\bar{w}_t = \gamma \bar{w}_{t-1} + (1-\gamma)\, w_t^*$$

$$w_t^* = (1-\theta^w)\omega + \theta^w w_{t-1}^* \frac{\bar{w}_{t-1}}{\bar{w}_{t-2}}$$

Contracted wages affect matches being created in the current period as well as previous matches of jobs that have survived from the previous period.[85]

These features complicate the problem, and to keep it tractable we assume contracts are allocated to workers and firms randomly every period. Random allocation of contracts ensures the firm not only hires the average worker looking for a job, it also hires and employs the "average contract". Nominal rigidity only affects the decisions of the firm via the average wage which is taken as given. Since the firm hires the average job searcher and pays the average contract, wage payments by the firm contain the average wage $\bar{w}_t$. Current profits are written in the same way as before and we get the first order condition for employment. As for the worker, the participation decision is also a function of the average contract on offer in the labor market, as we assume the worker cannot choose ex-ante any features of the employment she might get. We assume also that hours are a function of the average wage.

5.5.2 Bargaining[86]

The contract is the wage ω which appears here without a time subscript to help exposition. The surplus entering the bargaining equation is given by the value of agreement minus

[83] Adding an exogenous forward looking Phillips curve generates most of the properties of wages and employment we are interested in. However, it does not survive the Lucas critique. Christiano, L., Eichenbaum, M., and Trabant, M., (2016) make this point.

[84] The fraction of contracts $(1-\gamma)(1-\theta^w)$ adjusts by setting the wage equal to the average of the contracts updated last period adjusted for lagged wage inflation.

[85] Without rigidity $\bar{w}_t = \omega$. This is also a feature of the long run or of the structural model.

[86] Peter Bache designed the bargaining problem.

the value of disagreement in the bargaining game. Disagreement is an out of equilibrium event and is never observed. In the case of the union representing workers this surplus $S_t(\omega)$ is measured over all contracts being negociated $n_t \bar{\rho}_t \bar{h}_t (1-\gamma)(1-\theta^w)$, and obeys[87]

$$S_t(\omega) = (1-\bar{\tau}_t)\,\omega n_t \bar{\rho}_t \bar{h}_t (1-\gamma)(1-\theta^w) + \beta_{t+1}\gamma S_{t+1}(\omega)$$

where $\bar{\tau}_t$ is the weighted average of the personal income tax rate. This equation is proportional to the wage being bargained over so that it can be written $S_t(\omega) = \omega(1-\gamma)(1-\theta^w) \times \tilde{S}_t^+$ with

$$\tilde{S}_t^+ = (1-\bar{\tau}_t)\, n_t \bar{\rho}_t \bar{h}_t + \beta\gamma \tilde{S}_{t+1}^+$$

The derivative of the worker-side value with respect to ω is then given by $\partial S_t(\omega)/\partial\omega = (1-\gamma)(1-\theta^w)\tilde{S}_t^+$, which will prove useful below. This derivative disregards the contribution of ω to average hours and employment. Large monopoly unions could be assumed to internalize these effects. This myopia assumption is further discussed in the appendix on the bargaining problem.[88]

On the firm side the surplus aggregates all sectors j, and obeys the Bellman equation

$$J_t(\omega) = (1-\tau_t^c)(1-\gamma)(1-\theta^w)\bar{\rho}_t \bar{h}_t \left[J_t^{0+} - \omega J_t^{0-}\right] + \tilde{\beta}_{t+1}\gamma J_{t+1}(\omega)$$

$$J_t^{0+} \equiv \sum_j P_{j,t}^L z_{j,t} \rho_{j,t}^w n_{j,t}, \quad J_t^{0-} \equiv \sum_j \rho_{j,t}^w n_{j,t}\left(1+\tau_t^L\left(1+r_{j,t}^{self}\right)\right)$$

and here we isolate the negative part of this equation,

$$\tilde{J}_t^- = (1-\tau_t^c)\bar{\rho}_t \bar{h}_t \left[J_t^{0-}\right] + \tilde{\beta}_{t+1}\gamma \tilde{J}_{t+1}^-$$

such that

$$\frac{\partial J_t(\omega)}{\partial\omega} = -(1-\gamma)(1-\theta^w)\tilde{J}_t^-$$

The positive part of the surplus does contain ω implicitly through the marginal product in the firm's first order condition, but this effect is again ignored.

$$\tilde{J}_t^+ = (1-\tau_t^c)\bar{\rho}_t \bar{h}_t \left[J_t^{0+}\right] + \tilde{\beta}_{t+1}\gamma \tilde{J}_{t+1}^+$$

The Nash optimality condition is then

$$\frac{1-\phi^{Barg}}{S_t}\frac{\partial S_t}{\partial\omega} + \frac{\phi^{Barg}}{J_t}\frac{\partial J_t}{\partial\omega} = \left(1-\phi^{Barg}\right)\frac{1}{\omega} - \phi^{Barg}\frac{\tilde{J}_t^-}{J_t^+ - \omega\tilde{J}_t^-} = 0$$

which simplifies to

$$\omega = \left(1-\phi^{Barg}\right)\frac{J_t^+}{\tilde{J}_t^-} \tag{5.6}$$

Note that we have assumed that migrant workers are represented on both sides of the bargaining table.

Finally, we collect some of the objects from this section and equate them to their descriptions in the code. This is contained in Table 5.

[87] See the appendix for the derivation of this equation. One notable feature is the absence of the unemployment benefit, which is a consequence of the specific way the outside option is defined in the bargaining game. Ljungqvist and Sargent, (2017) discuss more standard formulations of the bargaining problem.

[88] This also implies the Bellman equation is not actually linear in ω. We assume the agents solving the problem act as if that was the case.

Table 5.5: Labor market code names: Matching and Wage Bargaining

α	eMatching
$\overline{w}_t$	vhW[t]
γ	rLoenTraeghed
θ^w	rLoenIndeksering
$\frac{v_t}{n_t^{s,agg}}$	rOpslag2soeg[t]
w_t^*	vhWNy[t]
ω	vhWForhandlet[t]
ϕ^{Barg}	rLoenNash[t]
$\tilde{J}_t^+$	vVirkLoenPos
J_t^{0+}	vVirkLoenPos0
$\tilde{J}_t^-$	vVirkLoenNeg
J_t^{0-}	vVirkLoenNeg0

5.6 Summary

The labor market solves with six key equations, and these six are highlighted by being numbered in the text. The first order condition for search and the law of motion for household employment, the first order condition for vacancies, the definition of the job finding rate, the equilibrium condition that matched vacancies equal jobs found, and the Nash bargaining solution. All other objects - such as the hours decision, the equations for wage rigidity and the aggregation equations - are auxiliary objects. The appendices that follow discuss model details.

5.7 Appendix 1

Here we use a simplified proxy of the model to illustrate how the model solves and calibrates. One important object in the labor market is the marginal product of labor. In the model this marginal product is obtained as part of the overall input choice in a CES tree structure. In the example we use here we abstract from other inputs and assume for simplicity a production function $F(n) = An^{\alpha}$ and a static model with 100 % job destruction every period. The firm posts vacancies and profits are $\pi = F - wn - \kappa \times v$. We also assume a simplified household choice where utility is given by $U = (w-b)n - g(n)$.

5.7.1 How the model solves

There are six key equations in the model. The first order condition for search and the law of motion for household employment, the first order condition for vacancies, the definition of the job finding rate, the equilibrium condition that matched vacancies equal jobs found, and the Nash bargaining solution.

$$\begin{array}{ccc} w_t - b = \lambda s_t^{\eta} & \Longrightarrow & \text{Household f.o.c.} \\ n_t = x_t s_t & \Longrightarrow & \text{Law of motion} \\ \frac{\partial F(n_t)}{\partial n_t} = w_t + \frac{\kappa}{m_t} & \Longrightarrow & \text{Firm f.o.c.} \\ x_t = f(v_t/s_t) & \Longrightarrow & \text{Matching function/jf rate} \\ x_t s_t = m_t v_t & \Longrightarrow & \text{equilibrium} \\ w_t = (1-\phi)\,\frac{\partial F(n_t)}{\partial n_t} & \Longrightarrow & \text{Nash bargaining} \end{array}$$

Given parameters, this system solves for $(w_t, s_t, n_t, x_t, v_t, m_t)$. The household f.o.c. "determines" search s, the law of motion links search with employment n, the firm f.o.c. "determines" vacancies, the job finding rate "determines" itself, the equilibrium condition "determines" m, and the Nash condition "determines" the wage.

5.7.2 How the model calibrates

We first need to use the available data to find values for our parameters. The six equations above have six variables $(w_t, s_t, n_t, x_t, v_t, m_t)$. We have data on wages and employment. We also make use of a labor force variable in the data to obtain a measure of s through the relationship $LF = (1-\delta)\, n + s$. This leaves three variables (x_t, v_t, m_t) to be calibrated by three parameters.

We can describe how the system solves as follows. Given data on (w, b, n, s) the first two equations (household f.o.c. and law of motion) solve for (λ, x). We are left with four equations which we use to find variables (v_t, m_t) and parameters (ϕ, κ). The matching function then solves for v and after that the equilibrium condition solves for m. We are left with two equations

$$\begin{array}{ccc} \alpha A n_t^{\alpha-1} = w_t + \frac{\kappa}{m_t} & \Longrightarrow & \text{Firm f.o.c.} \\ w_t = (1-\phi)\,\alpha A n_t^{\alpha-1} & \Longrightarrow & \text{Nash bargaining} \end{array}$$

which solve for (ϕ, κ).

Note that this solution leaves the parameter A inside the production function free. So, we have six equations and, given data on (n, s, w, b), we calibrate this model by solving the six equations for (λ, ϕ, v, m, x) and then we have a choice of using one of (A, κ) as endogenous in the calibration process.

This degree of freedom arises because the equation determining the χ function

$$\chi n = \kappa v \qquad \Longrightarrow \quad \chi \text{ function.}$$

is unconstrained and determines χ endogenously. However, if we set a calibration value for the level of χ the system is exactly identified and also pins down the value of the technology parameter A. In the main model this parameter A is a level parameter inside the CES structure.

5.8 Appendix 2

5.8.1 Imposing the same age distribution on every firm

The result of the household's first order condition for participation/search is that employment will vary by age. However, we do not want to add the sectoral dimension to the disaggregated employment variable. In order to do this, when we solve the problem of the firm we do not solve endogenously for the age distribution of workers inside the different firms/sectors indexed by j. [89] Instead we impose exogenously that this distribution is the same across all firms in the economy. Preliminary evidence from register data on wage earners indicates that the average age of the labor force is independent of firm size and also uncorrelated with whether firms are reducing or expanding their employment.

We force the same distribution using the relationship

$$n_{a,t,j} \equiv \frac{n_{t,j}}{n_t} n_{a,t}$$

Given our assumptions, we never have to use the bigger object $n_{a,t,j}$, since on the production side the age distribution does not matter and so we only care about total employment inside the firm.

5.8.2 Different average wages across sectors

Although in our model both labor supply and demand are anonymous, resulting in all firms hiring the same average worker looking for a job, and employing the same average employed worker in the economy, we observe in the data that average wages differ across sectors. It is possible that this reflects the heterogeneity of workers employed in different sectors, a feature which is ruled out in our model. In order to match the data on both employment and average wage acoss sectors we need a reduced form mechanism that will allow us to do so without breaking the two sided anonymity of the labor market.

The mechanism described here attaches different productivities to workers working in different sectors, while the workers themselves are identical wherever they happen to work. A three sector example helps illustrate it. We first impose the identifying constraint which attaches a relative sectoral productivity factor ρ_t^i to sectoral employment, while keeping the total constant:

$$\rho_{1,t}^w n_t^1 + \rho_{2,t}^w n_t^2 + \rho_{3,t}^w n_t^3 = \sum_i n_t^i = n_t$$

Given this constraint, calculate the average wage per sector in the data and compute the ratios:

[89] Different sectors will move differently over the cycle. And age specific population does not move evenly over time which implies neither will the labor force. All firms from all sectors hire the "average job searcher" from the currently available pool in the macroeconomy. As firms from different sectors hire different amounts over time the age composition of labor inside firms across sectors will differ, while it is the same in all firms within a sector. Since keeping track of the age distribution within each firm/sector greatly increases the dimensionality of the model we impose that all firms in the economy have the same age distribution of their workforce. We also choose not to allow for differences in the job destruction rate across sectors arising from other factors.

$$\frac{\rho^w_{1,t}}{\rho^w_{3,t}} = \frac{\bar{w}^1_t}{\bar{w}^3_t} = \bar{w}^{13}_t, \quad \frac{\rho^w_{2,t}}{\rho^w_{3,t}} = \frac{\bar{w}^2_t}{\bar{w}^3_t} = \bar{w}^{23}_t$$

Note: the objects $\bar{w}^{ij}_t$ are data for the period where data exists, and are forecasts for the subsequent periods. They are an exogenous input into the model. This allows for the endogenous calculation of $\rho^w_{3,t}$:

$$\rho^w_{3,t} = \frac{n_t}{\bar{w}^{13}_t n^1_t + \bar{w}^{23}_t n^2_t + n^3_t}$$

and of course of the other two as well. During data years we use observed average wages and employment, and in the forecasting years we use a forecast of relative average wages to calculate the $\rho^w_{j,t}$. This mechanism preserves the search model. It is consistent with the randomness of matching. Which means the household problem is unaffected because of the initial identifying constraint. And it can be interpreted as a proxy for heterogeneity.

5.8.3 Bargaining agreement versus disagreement

Wages faced by firms and workers move in the spirit of Galí and Gertler (1999). The fraction of contracts on the bargaining table is $(1-\gamma)(1-\theta^w)$ while another fraction of contracts $(1-\gamma)\theta^w$ adjusts in a mechanical way setting the wage equal to the contracts updated last period adjusted for lagged wage growth. We have

$$\bar{w}_t = (1-\gamma)\, w^*_t + \gamma \bar{w}_{t-1}$$

$$w^*_t = (1-\theta^w)\omega + \theta^w w^*_{t-1} \frac{\bar{w}_{t-1}}{\bar{w}_{t-2}}$$

We need to consider the value generated when there is agreement in the Bargaining problem, V, as well as that when there is no agreement, W. It is of course the case that there is never disagreement in equilibrium.

In case of agreement, on the worker side the total gain of employment over unemployment generated by this contract obeys the following Bellman equation:[90]

$$V_t(\omega) = ((1-\tau_t)\,\omega\rho_t h_t - b_t)\, n_t\,(1-\gamma)\,(1-\theta^w)$$
$$+\beta\gamma V_{t+1}(\omega) + \beta\,(1-\gamma)\, M_{t+1}(\omega_{t+1})$$

where the continuation value of this gain contains the value of the next reincarnation of this entire problem if the contract is destroyed, M. The bargaining agents are rational and understand that the wage being agreed upon today will have an effect on all future alternatives. However, this continuation value will cancel out of the problem, which saves us from having to specify these alternative paths. We also make specific myopia assumptions to further simplify the problem. An additional simplification is the absence of the utility function from this surplus. The monopoly union cares only about wages, not about utility.

In case of disagreement the current gain is zero for the workers affected, and in the continuation value if this contract is not destroyed, the gain remains zero. If the contract is destroyed next period, which happens with probability $(1-\gamma)$, the problem resumes its normal course and so

$$W_t = \beta\gamma W_{t+1} + \beta\,(1-\gamma)\, M_{t+1}(\omega_{t+1})$$

[90] Although the bargaining problem used in the model does not include it explicitly, in the data the unemployment benefit is indexed by a factor of circa 0.8 to an average of a reference wage from periods t-2 and t-3, and total unemployment income received has a ceiling which affects around two thirds of all wage earners.

Now, the surplus actually entering the bargaining equation is given by the value of agreement minus the value of disagreement:

$$S_t(\omega) = V_t(\omega) - W_t = ((1-\tau_t)\,\omega\rho_t h_t - b_t)\, n_t (1-\gamma)(1-\theta^w) + \beta\gamma S_{t+1}(\omega)$$

so that the continuation value cancels out of the problem.

We now assume the solution to this Bellman equation is proportional to ω. One way to rationalize this is that this is how the negotiating union sees the surplus. The unions sitting at the bargaining table are the ones doing this algebra. First decompose the surplus into its positive and negative components $S_t = S_t^+(\omega) - S_t^-$ where

$$S_t^+(\omega) = \omega x_t + \beta\gamma S_{t+1}^+(\omega)$$

$$S_t^- = b_t n_t (1-\gamma)(1-\theta^w) + \beta\gamma S_{t+1}^-$$

and $x_t = (1-\tau_t)\,\rho_t h_t n_t (1-\gamma)(1-\theta^w)$. The negative part of the surplus does not depend on ω. We can write the positive Bellman equation as $S_t^+(\omega) = \omega \times \tilde{S}_t^+$ where

$$\tilde{S}_t^+ = x_t + \beta\gamma \tilde{S}_{t+1}^+$$

The derivative of the worker-side value with respect to ω is given by the infinite sequence

$$\frac{\partial S_t(\omega)}{\partial\omega} = x_t + \beta\gamma x_{t+1} + \beta\gamma\beta\gamma x_{t+2}... = \tilde{S}_t^+$$

This is where our assumptions become active. We have large unions aggregating preferences of all agents on their side of the market, and yet we work through the problem without internalizing the fact that hours, employment, and average wages $\bar{w}$ will respond to the wage currently being bargained.

On the firm side the same applies, yielding the following Bellman equation

$$J_t(\omega) = (1-\tau_t^c)\,[p_t F_L \xi_t - \omega]\, h_t\rho_t n_t (1-\gamma)(1-\theta^w) + \beta\gamma J_{t+1}(\omega)$$

and we can separate the two terms in this equation and extract ω to obtain

$$J_t^+ = (1-\tau_t^c)\,[p_t F_L \xi_t]\, h_t\rho_t n_t (1-\gamma)(1-\theta^w) + \beta\gamma J_{t+1}^+$$

$$\tilde{J}_t^- = y_t + \beta\gamma \tilde{J}_{t+1}^-$$

with $y_t = (1-\tau_t^c)\, h_t\rho_t n_t (1-\gamma)$ so that $J_t(\omega) = J_t^+ - \omega\tilde{J}_t^-$, and the derivative on the firm side is given by

$$-\frac{\partial J_t(\omega)}{\partial\omega} = y_t + \beta\gamma y_{t+1} + ... = \tilde{J}_t^-$$

The general Nash optimality condition is

$$\frac{\phi^{Barg}}{J_t}\frac{\partial J_t}{\partial\omega} + \frac{1-\phi^{Barg}}{S_t}\frac{\partial S_t}{\partial\omega} = 0$$

and replacing the objects above we obtain

$$\phi^{Barg}\frac{\tilde{J}_t^-}{J_t^+ - \omega\tilde{J}_t^-} = \left(1-\phi^{Barg}\right)\frac{\tilde{S}_t^+}{\omega\tilde{S}_t^+ - S_t^-}$$

or

$$\omega = \frac{S_t^-\phi^{Barg}\tilde{J}_t^- + J_t^+\tilde{S}_t^+\left(1-\phi^{Barg}\right)}{\tilde{J}_t^-\tilde{S}_t^+} = \phi^{Barg}\frac{S_t^-}{\tilde{S}_t^+} + \left(1-\phi^{Barg}\right)\frac{J_t^+}{\tilde{J}_t^-}$$

This expression is our "supply curve" and closes the model, which solves for the wage per hour per unit of productivity, and for employment, unemployment and hours.[91]

[91] There is a significant degree of arbitrariness in the determination of the bargaining wage as any wage interior to the admissible equilibrium range is a solution, and not much is known regarding what affects the wage as it moves within this range. See Blanchard and Gali (2008).

5.8.4 Worker surplus used in MAKRO

We define the outcome from not agreement as implying the worker is not entitled to the unemployment benefit. We obtain

$$W_t = -b_t n_t (1-\gamma)(1-\theta^w) + \beta\gamma W_{t+1} + \beta(1-\gamma) M_{t+1}(\omega_{t+1})$$

in which case

$$S_t(\omega) = (1-\tau_t)\,\omega\rho_t h_t n_t (1-\gamma)(1-\theta^w) + \beta\gamma S_{t+1}(\omega)$$

so that here

$$S_t^- = 0$$

and therefore

$$\omega = \left(1-\phi^{Barg}\right)\frac{J_t^+}{\tilde{J}_t^-}$$

6 Exports

Most exported goods are produced at home. A small fraction of exported goods consists of imported goods which are immediately exported back. These are goods in transit and are treated separately. Any valued added generated by the transit process is of course a part of exports but this is separated from the valuation of the goods imported and exported back.

In MAKRO we organize the data into five export "items" or "components": energy, goods, sea transport, services and tourism.[92] Table 1 shows the evolution of total exports of domestically produced goods as well as of the individual items over time. The Danish economy is significantly more open today than it was a few decades ago. In 2017 the bulk (54%) of exports consists of goods while tourism and energy make up around 8% of the total.

Exports from domestic production rise from 30% of GDP in 1980 to 46% of GDP in 2017 while exports from imports shown in Table 2 rise from 2.2% of GDP in 1980 to 8.6% of GDP in 2017. This observed trend in the available data implies we need to forecast its evolution into the future. We do this by forecasting elements of the model governing the demand for exports.

6.1 Exports of Domestically produced goods.

6.1.1 Demand for the five export components

The demand curve for each of the five export goods draws both on the Armington model - Armington (1969), Anderson (1979) - and on the Gravity equations from Anderson and Van Wincoop (2003). Using a single Armington type equation to describe the demand for exports of a particular good in a small open economy model carries a number of assumptions regarding aggregation of demand curves originating in different countries since aggregation is linear while the demand function is nonlinear. As the demand for exports is exogenous to the model the specification we use is, while grounded in theory, empirically pragmatic.

There are five of these equations, one for each export good (subscript x) which is directly sourced from domestic production (superscript y), $X^{y}_{x,t}$:[93]

$$X^{y}_{x,t} = \mu^{Xy}_{x,t} Q^{XM}_{x,t} q^{Scale}_{t} \left(R^{X}_{x,t}\right)^{-\eta^{X}_{x}}$$

As in the CES demand problem from Anderson (1979) exports relate to price ratios $R^{X}_{x,t}$ with a given elasticity η^{X}_{x}, and market size $Q^{XM}_{x,t}$ approximates for the aggregate income of foreign consumers demanding the good. In this way this equation is similar to any of the CES demand curves originating from our househodls or firms.

Scale

The scale variable q^{Scale}_{t} measures the growth of the domestic economy. It adds to the demand for exports an element of "supply generating its own demand". We use a dynamic construction

$$q^{Scale}_{t} = \left(L^{p}_{t}\right)^{1-\alpha} \left(q^{Scale}_{t-1}\right)^{\alpha}$$

[92] Index set $x = \{xEne, xVar, xSoe, xTje, xTur\}$ where the prefix x stands for export.

[93] Exports sourced from domestic production have a superscript y, while goods imported to be exported carry the superscript m. As these objects enter the input-output aggregation algebra where objects are assigned the superscript IO, the overall superscript will be IOy or IOm.

with $\alpha = 0.7$ and where L_t^p is the structural level of the sum of all private sector effective labor input.[94] Introducing q_t^{Scale} is suggested by the solution to the gravity model derived in Anderson and Van Wincoop (2003). The demand curve that results from maximizing a CES objective function is a partial equilibrium object that cannot contain the income of the supply side. However, the solution to the trade model - Eaton and Kortum (2002), Melitz (2003) - will contain it in some way. As we are not solving for world trade equilibrium we include this feature in a reduced form way. Technically what results is not a demand curve stricto sensu but rather a description of what drives the quantity exported.

Market Size

The variable $Q_{x,t}^{XM}$ is a dynamic construction which uses the size of the export market $q_{x,t}^{XM}$ taken from ADAM. With $\varphi = 0.29$ we have[95]

$$Q_{x,t}^{XM} = \left(q_{x,t}^{XM}\right)^{1-\varphi} \left(Q_{x,t-1}^{XM}\right)^{\varphi}$$

Price Ratio

The object $R_{x,t}^{X}$ is a relative price ratio construction which uses the sector's export price, $P_{x,t}^{Xy}$, relative to its export competing price, $P_{x,t}^{XF}$. This foreign price $P_{x,t}^{XF}$ is the "world price" in the respective export market and is first taken from the data and then forecasted. It is exogenous to the model. The domestic export price $P_{x,t}^{Xy}$ reflects the way the export good X is sourced from the nine domestic production sectors, and that composition is summarized by the factors/parameters $u_{x,s,t}^{IOy}$. We write

$$p_{x,t}^{Xy} = \sum_{s \in y(x)} u_{x,s,t}^{IOy} p_{x,s,t}^{IOy}$$

where the set $y(x) \equiv d_1^{IOy}(x,s,t)$ denotes the subset of the nine production sectors involved in the making of the particular export good x. This equation is further conditioned by the set $d_1^{Xy}(x,t)$ which defines the subset of export goods x the equation describes (in this case the full set, all five export goods). We describe the parameters $u_{x,s,t}^{IOy}$ below.

The object $R_{x,t}^{X}$is constructed as follows:

$$R_{x,t}^{X} = \frac{P_{x,t}^{Xy}}{P_{x,t}^{XF}} - \gamma R_{x,t}^{X} \left[\frac{R_{x,t}^{X}}{R_{x,t-1}^{X}} - 1 \right] \frac{R_{x,t}^{X}}{R_{x,t-1}^{X}}$$

$$+\beta_{t+1} \gamma R_{x,t+1}^{X} \left[\frac{R_{x,t+1}^{X}}{R_{x,t}^{X}} - 1 \right] \left[\frac{R_{x,t+1}^{X}}{R_{x,t}^{X}} \right] \left[\frac{X_{x,t+1}^{y}}{X_{x,t}^{y}} \right] \frac{P_{t+1}^{XF}}{P_{t}^{XF}}$$

with $\gamma = 12$ and β being a discount factor. This construction brings a forward looking element into the demand for exports, and is derived by considering a foreign firm which buys from Denmark and then solves an optimal price setting problem when selling to its consumers in a market characterized by monopolistic competition and price rigidity.

We derive explictly this equation at the end of this chapter. The usefulness of this construction is that it allows for changes in the prices $P_{x,t}^{Xy}$ or $P_{x,t}^{XF}$ to affect the quantity exported in a more dynamic and nuanced way than the effect obtained through the standard demand expression.

[94]In the code $\alpha$is labeled 'rXSkalaTraeghed' and the scale variable is indexed x for generality, $q_{x,t}^{Scale}$, but in its current implementation it is independent of it.

[95]In the code $\varphi$is labeled 'rXTraeghed'.

Export Elasticity

The export elasticity, η_s^X, is currently set to 5 for all goods as in DREAM, with the exception of energy where it has value 5.6. This export elasticity is a key parameter in MAKRO as it is the source of overall aggregate diminishing returns which allows the model to have a solution. The empirical work leading to the elasticity values used is detailed elsewhere.

Level parameters

The object $\mu_{x,t}^{Xy}$ is a parameter which accounts for the long run possibility that exports of a given type grow or decline even when the size of the export market or relative prices do not change. In equilibrium this parameter $\mu_{x,t}^{Xy}$ measures the average trend of a ratio such as, for example, the amount of oil Denmark exports, $X_{x,t}$ where the subscript x would be oil, relative to the entire world oil production $Q_{x,t}^{XM}$. This parameter $\mu_{x,t}^{Xy}$ is measured on available data and then forecast. The actual value of the parameter uses the forecast series $\mu_{x,t}^{ARIMA}$ as follows, where T^* is the last date of available data:

$$\mu_{x,t}^{Xy}|_{t>T^*} = \mu_{x,t}^{ARIMA} \times \frac{\mu_{x,T^*}^{Xy}}{\mu_{x,T^*}^{ARIMA}}$$

where $\mu_{x,T^*}^{Xy} \neq \mu_{x,T^*}^{ARIMA}$ because the ARIMA variable is the projection while the other is the actual value.[96]

Obtaining the ARIMA

The parameters $\mu_{x,t}^{Xy}$ in the export demand equations are obtained in the data years from taking the export demand equation

$$\frac{X_{x,t}^{y}}{Q_{x,t}^{XM} q_{x,t}^{Scale} \left(R_{x,t}^{X}\right)^{-\eta_x^X}} = \mu_{x,t}^{Xy}$$

Since we have the elasticity η estimated elsewhere and all other objects are data, we obtain the time series of $\mu_{x,t}^{Xy}$ for the data years. With this time series we then run an ARIMA forecast to generate the future values of this variable.

6.1.2 Composition

Here we look at the composition of the quantity objects $X_{x,t}^{y}$ and the price objects $P_{x,t}^{Xy}$ in terms of the production sectors s they are sourced from. Exports are organized and classified differently from both consumption and production. The five exports differ from the five goods consumed by domestic households in that they are sourced differently from the nine domestic production sectors. The export groups are formed on the basis of the Standard International Trade Classification (SITC) of foreign trade, the consumption groups are formed on the basis of the consumption groups defined in the National Accounts (NR, National Regnskab), and the production groups are formed from the industry classification in NR. The Input-Output system keeps track of all flows from industries to export and consumption groups. An example of the difference in classification is the mapping from the production of sea transport services into consumption and into export items. Sea transport is now its own separate export item whereas in the domestic five-good consumption classification it is included in all services consumed by the household.

[96] In the ordinary least squares example the projection of the dependent variable Y on the space of regressors X is given by $\hat{\beta}_{ols}X$.

Obtaining quantities and prices starts with an allocation of *nominal* quantities. We use an auxiliary factor $f^y(x,s,t)$ which is a nominal share of the total. Real quantities are then induced from the nominal allocation. Consider an example with 2 production sectors generating one arbitrary export good x, $X_{x,t}$. Given parameters $u_{x,s,t}$ and production prices $p_{x,s,t}$ we construct an export good price linearly

$$p_{x,t} = u_{x,1,t}p_{x,1,t} + u_{x,2,t}p_{x,2,t}$$

and then define the quantity sourced for this export good x from, for example, production sector one, $q_{x,1,t}$, from the relationship

$$p_{x,1,t}q_{x,1,t} = f_{x,1,t}p_{x,t}X_{x,t} \equiv \frac{u_{x,1,t}p_{x,1,t}}{p_{x,t}}p_{x,t}X_{x,t}$$

The sum of factors $f_{x,1,t} + f_{x,2,t}$ yields *exactly* 1. Individual quantity levels $q_{x,s,t}$ are induced by the aggregate quantity $X_{x,t}$ which is determined elsewhere. In fact this equation reduces to the exogenous proportion

$$q_{x,1,t} = u_{x,1,t}X_{x,t}$$

One important detail to note here is that the production sector prices have an export good index x attached, $p_{x,s,t}$. The reason is the presence of export duties which are allocated at this level in the model. The firm producing the goods that go into the making of an export good does not receive these taxes, but they are paid by the - in this case foreign - consumer.

In the terminology of the code the f factors shown in Table 3 are:

$$f^y(x,s,t) \equiv u^{IOy}_{x,s,t}\frac{p^{IOy}_{x,s,t}}{p^{Xy}_{x,t}}$$

so that

$$v^{IOy}_{x,s,t} \equiv f^y(x,s,t)\,v^{Xy}_{x,t} \equiv f^y(x,s,t)\,\underbrace{p^{Xy}_{x,t}q^{Xy}_{x,t}}_{\text{value } v^{Xy}_{x,t}} \equiv u^{IOy}_{x,s,t}p^{IOy}_{x,s,t}q^{Xy}_{x,t}$$

and

$$\sum_s v^{IOy}_{x,s,t} = v^{Xy}_{x,t}\underbrace{\sum_{s\in y(x)} f^y(x,s,t)}_{1} = v^{Xy}_{x,t}$$

Around 65% of domestically produced energy exports are sourced from the domestic energy sector itself while around 35% are sourced from the domestic extraction sector.

Tourism is modeled differently.

6.1.3 Tourism

One of the export components is the spending of foreign tourists in Denmark. Table 1 shows this item to be relatively small and without trend at around 2% of GDP. Unlike the other four export items, in the case of tourism the mapping from the demand equation to the nine sector production organization is an indirect one, and occurs via the decomposition used for household consumption. Tourists consume (a subset of) the same objects as domestic households do, and therefore their demand then branches down to the nine sector domestic production in that same way.

In order to achieve this we need an allocation from total tourist demand $X_{'xTur',t}$ to a consumption classification c. This occurs through the following equation:

$$\underbrace{f_c^{PCT} P_{c,t-1}^C}_{P_{c,t-1}^{CTourist}} C_{c,t}^{Tourist} = \mu_{c,t}^{CTourist} P_{'xTur',t-1}^X X_{'xTur',t}$$

The tourist consumption division into consumption groups is not available from national accounts data. It is instead imputed using ADAM's equation for the price index of tourism exports.[97] This equation has lagged prices because the data is constructed using a chain index approach for the quantity $X_{'xTur',t}$. [98]

The result of this construction is that we always have

$$\sum_c P_{c,t}^{CTourist} C_{c,t}^{Tourist} = P_{'xTur',t}^X X_{'xTur',t}$$

Finally, the term f_c^{PCT} captures the fact that consumption of tourists has a different deflator than consumption of locals, otherwise the price $P_{c,t-1}^{CTourist}$ would equal the domestic consumption price. This correction is small, $f_c^{PCT} = 1.022021$. The distribution of tourist consumption into goods, energy and services used is the one from ADAM. This distribution does not yield exactly the correct aggregate price index for tourist consumption in Denmark. One reason may be that tourists have a different composition of consumption of goods, energy and services than the resident population. There is therefore a need to correct the deflator - this is done with a uniform factor on all items which remains constant into the future.

6.2 Imports for Export

Regarding quantities that are imported and exported back, we can see their relative weight in GDP in Table 2. These amount to 8.7% of GDP in 2017, which contrasts with the 46% of GDP commanded by exports from domestic production.

For 2017 data there are three instances where imported goods are then immediately exported. Goods purchased from foreign manufacturing, energy purchased from foreign energy producers, and sea transport purchased from foreign service sector providers. Table 3 shows the value ratio of these purchases compared to the same ones sourced from domestic production $v_{x,s,t}^{IOm} / \left(v_{x,s,t}^{IOy} + v_{x,s,t}^{IOm} \right)$.

Note that, although many elements are empty and columns are therefore not reported, Table 3 maps 9 production sectors into 5 export groups. In 2017, 18.4% of the total value of energy exports *sourced from the energy production sector,* arise from imported energy which is exported back immediately. From other data this means 18.4% of 65.1% of total energy exports come from imported energy. This quantity is modeled similarly to the main equation above but it is static:

$$X_{x,t}^m = \mu_{x,t}^{Xm} Q_{x,t}^{XM} \left(P_{x,t}^{Xm,Rel} \right)^{\eta_x^X}$$

and note that it has the same elasticity η_x^X and the same export market variable $Q_{x,t}^{XM}$ as above.

The relative price reflects taxation of these goods in transit and is given by

[97] We have an estimate for $\mu_{c,t}^{CTourist}$ based on ADAM's weights for calculating $P_{'xTur',t-1}^X$. This price is called "pet" in ADAM.

[98] To capture the national accounting method, $X_{'xTur',t}$ must be a chain index of $C_{c,t}^{Tourist}$ in data. Typically, however, we write it as a CES index with a correction factor - where the correction factor captures the difference between CES and chain index.

$$
\begin{aligned}
P_{x,t}^{Xm,Rel} &= \sum_{s\in d_1^{IOm}(x,s,t)} \frac{u_{x,s,t}^{IOm} p_{s,t}^{M}}{p_{x,t}^{Xm}} \\
&= \sum_{s\in d_1^{IOm}(x,s,t)} u_{x,s,t}^{IOm} p_{s,t}^{M} \times \underbrace{\left[\sum_{s\in d_1^{IOm}(x,s,t)} u_{x,s,t}^{IOm} p_{x,s,t}^{IOm}\right]}_{p_{x,t}^{Xm}}^{-1} \\
&= \sum_{s\in d_1^{IOm}(x,s,t)} u_{x,s,t}^{IOm} p_{s,t}^{M} \times \left[\sum_{s\in d_1^{IOm}(x,s,t)} u_{x,s,t}^{IOm} \underbrace{\left(1+\tau_{x,s,t}^{IOm}\right) p_{s,t}^{M}}_{p_{x,s,t}^{IOm}}\right]^{-1}
\end{aligned}
$$

and this equation exists only in the conditioning set $d_1^{Xm}(x,t)$.

While at first glance the numerator and denominator seem very similar, the tax factor $\tau_{x,s,t}^{IOm}$ can be significant for some (x,s) pairs, and also, for some of these pairs it does vary over time.

The level parameter.

The value of $\mu_{x,t}^{Xm}$is again calculated from the data. Given the construction of the relative price we only have to invert the relationship to obtain

$$
\frac{X_{x,t}^{m}}{Q_{x,t}^{XM}\left(P_{x,t}^{Xm,Rel}\right)^{\eta_x^X}} = \mu_{x,t}^{Xm}
$$

and then forecast the time series in the usual way. Obtaining the relative price requires having values for the lower level parameters.

6.3 Derivation of the dynamic price ratio

Consider the demand for exports given by the Armington equation

$$
X_t = \mu Q_t \left(\frac{\overline{P}_t^X}{P_t^F}\right)^{-E}
$$

where $\overline{P}_t^X$ is the price of exports faced by foreign consumers and P_t^F is the price of competing foreign products. We now introduce intermediaries (located abroad) that buy exports at price P_t^X and sell to their consumers at the price $\overline{P}_t^X$. With free entry and no additional costs, $\overline{P}_t^X = P_t^X$. This is the baseline model without rigidity.

Assume now that intermediaries face monopolistic competition and have an adjustment cost of setting their price. The problem is

$$
\overline{P}_t^X = \arg\max_{p_{i,t}} \left[p_{i,t} - P_t^X\right] X_t \left[\frac{p_{i,t}}{\overline{P}_t^X}\right]^{-\eta} - \left[g_t + \beta_{t+1} g_{t+1}\right]
$$

The first order condition of each intermediary is

$$
X_t \left[\frac{p_{i,t}}{\overline{P}_t^X}\right]^{-\eta} \left(1 - \eta\left[p_{i,t} - P_t^X\right]\frac{1}{p_{i,t}}\right) - \left[\frac{\partial g_t}{\partial p_{i,t}} + \beta \frac{\partial g_{t+1}}{\partial p_{i,t}}\right] = 0
$$

and imposing symmetric equilibrium $p_{i,t} = \overline{P}_t^X$ this writes

$$\overline{P}_t^X = \frac{\eta}{\eta-1} P_t^X - \frac{1}{\eta-1} \frac{\overline{P}_t^X}{X_t} \left[\frac{\partial g_t}{\partial p_{i,t}} + \beta \frac{\partial g_{t+1}}{\partial p_{i,t}} \right]$$

and this expression shows that one can have zero (or close to zero) markups (the difference $\overline{P}_t^X - P_t^X$) without having perfect competition. On the other hand, as $\eta \to +\infty$ we do have that $\overline{P}_t^X = \to P_t^X$ (perfect competition implies zero markups).

It is useful to define the adjustment cost as follows

$$g_t \equiv \frac{\gamma}{2} \left[\frac{p_{i,t}}{P_t^F} \frac{P_{t-1}^F}{p_{i,t-1}} - 1 \right]^2 \overline{P}_t^X X_t$$

Defining the price ratio $R_t = \overline{P}_t^X / P_t^F$ and imposing symmetry yields

$$R_t = \frac{\eta}{\eta-1} \frac{P_t^X}{P_t^F} - \frac{\gamma}{\eta-1} R_t \left[\frac{R_t}{R_{t-1}} - 1 \right] \frac{R_t}{R_{t-1}}$$

$$+\beta_{t+1} \frac{\gamma}{\eta-1} R_{t+1} \left[\frac{R_{t+1}}{R_t} - 1 \right] \left[\frac{R_{t+1}}{R_t} \right] \frac{X_{t+1}}{X_t} \frac{P_{t+1}^F}{P_t^F}$$

Finally multiply by the factor $\frac{\eta-1}{\eta}$ to obtain the expression used above:

$$R_t \frac{\eta-1}{\eta} \equiv \hat{R}_t = \frac{P_t^X}{P_t^F} - \underbrace{\frac{\gamma}{\eta-1}}_{\hat{\gamma}} \hat{R}_t \left[\frac{\hat{R}_t}{\hat{R}_{t-1}} - 1 \right] \frac{\hat{R}_t}{\hat{R}_{t-1}}$$

$$+\beta_{t+1} \hat{\gamma} \hat{R}_{t+1} \left[\frac{\hat{R}_{t+1}}{\hat{R}_t} - 1 \right] \left[\frac{\hat{R}_{t+1}}{\hat{R}_t} \right] \frac{X_{t+1}}{X_t} \frac{P_{t+1}^F}{P_t^F}$$

Having derived the dynamic relationship determining the price ratio some comments are in order. Going back to the initial demand function we can see that using the $\hat{R}_t$ ratio is equivalent to redefining the demand parameter μ

$$X_t = \underbrace{\left[\frac{\eta}{\eta-1} \right]^{-E} \mu}_{\hat{\mu}} Q_t \left(\hat{R}_t \right)^{-E}$$

$$\hat{\mu} = \left[\frac{\eta}{\eta-1} \right]^{-E} \mu$$

as well as redefining the adjustment cost parameter

$$\hat{\gamma} = \frac{\gamma}{\eta-1}$$

The value of η is not separately identifiable which implies we cannot judge how competitive the intermediary price setting market is. Irrespective, what matters is that the problem is well specified for any finite value of $\eta > 0$, and for any large value of η we have that $\hat{\mu} \approx \mu$ and that, as it should, the price ratio approaches its spot value $\hat{R}_t \to P_t^X / P_t^F$.

Table 6.1: Exports in GDP*

Year	Energy	Goods	Sea Trans	Services	Tourism	Total
1980	0.009965	0.209866	0.030159	0.033851	0.019014	0.302856
1990	0.009869	0.220335	0.027247	0.051062	0.023638	0.332150
2000	0.026213	0.224690	0.066215	0.060894	0.021287	0.399299
2010	0.026346	0.223389	0.091160	0.072003	0.018959	0.431858
2017	0.014667	0.252829	0.082856	0.091583	0.023563	0.465498
		Relative Contribution of each Export Item				
Year	Energy	Goods	Sea Trans	Services	Tourism	Total
2017	0.0315	0.5431	0.1780	0.1967	0.0506	1

*Nominal current price ratios of exports to GDP, $p_{i,t}^{Xy} q_{i,t}^{Xy} / p_t^{GDP} Q_t^{GDP}$

The value of exports of domestically produced goods is labelled in the code $v_{x,t}^{IOy}$.

Table 6.2: Exports of Imported Goods in GDP*

Year	Energy	Goods	Sea Trans	Services	Total
1980	0.000316	0.021857			0.022173
1990	0.000763	0.031302		0.000016	0.032081
2000	0.001472	0.048124			0.049596
2010	0.004078	0.060029	0.007360	0.001905	0.073372
2017	0.002155	0.079128	0.005320		0.086603

*Nominal current price ratios of exports to GDP, $p_{i,t}^{Xm} q_{i,t}^{Xm} / p_t^{GDP} Q_t^{GDP}$

Table 6.3: Relative Value of Imported Exports

	Production Sectors		
	Man	Ser	Ene
X = Goods	0.30268		
X = Energy			0.18421
X = Sea T.		0.69413	

2017 data. Value ratio. $v_{x,s,t}^{IOm} / \left(v_{x,s,t}^{IOy} + v_{x,s,t}^{IOm} \right)$.
$v_{x,s,t}^{IOm}$ denotes the value of goods imported and immediately exported back.

7 Government

This chapter details government revenues and expenditures. A number of items on the expenditure side of the balance sheet are exogenous, or obey exogenous relationships for example to population or GDP. On the revenue side the same is true as much of this side of the balance sheet amounts to the determination of the realized average tax rate on a particular item.

A couple of simple relationships are useful to put forward here. First, the government budget is the primary budget plus net interest income,

$$Bdg_t = PrBdg_t + Netr_t^y$$

and the primary budget is the net of revenues minus expenditures.

$$PrBdg_t = REV_t - EXP_t$$

Revenue and expenditure are described in the netx two sections. Net interest income is described after that. After detailing the balance sheet, we define the structural budget balance and the fiscal sustainability indicator.

7.1 Revenue

Government revenue is given by the sum of direct taxation, indirect taxation and other government revenues:

$$REV_t = T_t = T_t^{Direct} + T_t^{Indirect} + T_t^{Other}$$

Direct taxes consist mainly of general income taxation, with corporate taxation, taxation on housing, and other taxes contributing smaller amounts. Direct taxes make up around 60% of total tax revenues and are described in 7.1.1 which covers many aspects of the Danish income tax system. Indirect taxes consist mainly of duties, VAT and production taxes, and are described in 7.1.3. Indirect taxes make up around 30% of tax revenues. The remaining taxation is described in 7.1.4.

Regarding notation, the letter y stands for income and appears in different objects, tax rates are denoted by τ, and tax revenues are given by the capital letter T. For example, the sector specific corporate tax rate is called $\tau_{s,t}^{Corp}$ and the total revenue is called $T_{s,t}^{Corp}$. Tax rates in the text, τ, correspond to tax rates in the code $t \times f$, where f is an adjustment variable to fit the data. These adjustments help match observed average tax rates, given the rate determined in the tax law. The adjustment factor is sometimes unnecessary and set to 1.[99]
Where applicable, variables such as T_t^{Income} represent sums over all cohorts, while corresponding variables with an age subscript, $T_{a,t}^{Income}$, represent cohort averages. The two variables are related by $T_t^{Income} = \sum_a N_{a,t} T_{a,t}^{Income}$ where $N_{a,t}$ is population.

7.1.1 Direct taxation

Direct taxation is modeled closely after the Danish income tax law.[100] Economic and demographic movements affect the tax burden such that the relationship between direct

[99] The appendix contains a table with all government revenues and expenditures; their name, value, how they are corrected regarding structural level, and which ADAM variable they are correspond to.

[100] See http://www.skm.dk/skattetal/beregning/skatteberegning/skatteberegning-hovedtraekkene-i-personbeskatningen-2017

taxation and the total income level is not constant, and therefore we need a flexible modeling of the tax system.

Direct taxation consists of income taxes T_t^{Income}, labor market contributions (AM bidrag) T_t^{AM}, other personal income taxation, T_t^{OPers}, weight duties on cars, T_t^{Weight}, corporate taxation, T_t^{Corp}, taxation on the return on investments in pension funds, T_t^{PAL}, and the contribution to the publicly owned media, T_t^{Media}:[101]

$$\begin{aligned} T_t^{Direct} = & \quad T_t^{Income} + T_t^{AM} + T_t^{OPers} \\ & + T_t^{Weight} + T_t^{Corp} + T_t^{PAL} + T_t^{Media} \end{aligned}$$

The income tax has a number of components. Furthermore, as income varies over the life cycle so does the income tax revenue. We have then

$$\begin{aligned} T_{a,t}^{Income} = & \quad T_{a,t}^{Bot} + T_{a,t}^{Top} + T_{a,t}^{Muncipal} + T_{a,t}^{Property} \\ & + T_{a,t}^{Stock} + T_{a,t}^{Business} + T_{a,t}^{Deceased} \end{aligned}$$

where the first two components divide income in two groups with revenues for bottom and top income taxation. The next two items are local (municipal) income taxes and property taxes. The last three items are taxes on capital income from stocks, taxes on small businesses that do not pay corporate tax, and taxes on the deceased, as they can still have income subject to taxation in the year they die.

The revenue from bottom income taxation is given by:

$$T_{a,t}^{Bot} = \tau_{a,t}^{Bot} \left[y_{a,t}^{Personal} + y_{a,t}^{NetCap^+} - y_{a,t}^{PA} \right]$$

with bottom tax rate τ_t^{Bot} , and is based on personal income, $y_{a,t}^{Personal}$, net income from bonds and deposits $y_{a,t}^{NetCap^+}$ which for this tax purpose is conditional on being positive and above a certain level, and a personal allowance $y_{a,t}^{PA}$, which lowers the tax burden.

The revenue from top income taxation is given by:

$$T_{a,t}^{Top} = \tau_t^{Top} \cdot \left[y_{a,t}^{Personal} + y_{a,t}^{NetCap^+} \right] \cdot \alpha_{a,t}$$

and here only income above a certain threshold is taxed at this tax rate. The object $\alpha_{a,t}$ controls for this fraction of income so that $T_{a,t}^{Top}$ fits the data.

The municipal tax is given by:

$$T_{a,t}^{Muncipal} = \tau_{a,t}^{Muncipal} \cdot \left[y_{a,t}^{Taxable} - y_{a,t}^{PA} \right]$$

The municipal taxation is based on taxable income with the personal allowance subtracted. Taxable income and personal income are defined below.

The taxation on property follows the value of the primo stock of privately owned housing, $H_{a-1,t-1}^{Private}$:

$$T_{a,t}^{Property} = \tau_t^{Property} \cdot H_{a-1,t-1}^{Private}$$

[101] AM Bidrag is a tax of 8%, which all employees and the self-employed must pay each month on their wages. Employers ensure that the labor market contribution is automatically deducted from salary after ATP and any own pension contribution have been deducted, after which the other taxes are deducted.

where $\tau_t^{Property}$ is an implicitly calculated tax rate.[102]

The taxation on income generated by financial stocks is given by:

$$T_{a,t}^{Stock} = \tau_t^{Stock} \cdot \left(r_{div,t}^{Foreign} \cdot S_{a-1,t-1}^{Foreign} + r_{div,t}^{Home} \cdot S_{a-1,t-1}^{Home} + C_{a,t}^{Gains} \right)$$

where τ_t^{Stock} is the implicit tax rate. Both dividends and realized capital gains are subject to taxation on stock income.[103] The realized capital gains are modelled as a slow moving average of the actual stock:

$$C_{a,t}^{Gains} = 0.95 \cdot C_{a-1,t-1}^{Gains} + 0.05 \cdot \left[r_{cgains,t}^{Foreign} \cdot S_{a-1,t-1}^{Foreign} + r_{cgains,t}^{Home} \cdot S_{a-1,t-1}^{Home} \right]$$

so that capital gains are gradually taxed with an average realization time of 20 years. The objects $\left(S_{a-1,t-1}^{Foreign}, S_{a-1,t-1}^{Home} \right)$ are part of the household portfolio. $r_{cgains,t}^{Foreign}$ is given by an exogenous required rate of return and an exogenous foreign dividend rate. $r_{cgains,t}^{Home}$ is an endogenous object as it depends on the value of the firm (plus an exogenous dividend rate).

The business tax follows earnings before taxes, EBT_t, with an implicit tax rate, $t_t^{Business}$. It is distributed among cohorts according to their wage income assuming that business income follows wage income:

$$T_t^{Business} = \sum_a T_{a,t}^{Business} N_{a,t} = \tau_t^{Business} \cdot EBT_t \cdot \sum_a \frac{n_{a,t}^e w_{a,t}}{\sum_a n_{a,t}^e w_{a,t}} = \tau_t^{Business} \cdot EBT_t$$

where $n_{a,t}^e$ denotes employment of cohort a in period t.[104]

Taxation on the deceased are primarily taxes on capital income of the deceased. It follows the base for taxation of stocks, $T_{a,t}^{Stocks}/\tau_t^{Stocks}$, and other capital income, $y_{a,t}^{NetCap^+}$, for those that do not survive into next period, $1 - s_{a,t}$:

$$T_{a,t}^{Death} = \tau_t^{Death} \cdot (1 - s_{a,t}) \cdot \left(\frac{T_{a,t}^{Stocks}}{t_t^{Stocks}} + y_{a,t}^{NetCap^+} \right)$$

The labor market contribution (AM Bidrag) is modeled as follows:

$$T_{a,t}^{AM} = \tau_t^{AM} \cdot \left[\frac{n_{a,t}^e w_{a,t}}{N_{a,t}} \right] \cdot \left[\frac{\sum_a n_{a,t}^e w_{a,t} - T_t^{CivilServants}}{\sum_a n_{a,t}^e w_{a,t}} \right]$$

It depends on wages per person (not per employee) adjusted for pension contributions to civil servants pensions and the tax rate.[105]

Direct taxation also contains other (residual) personal income taxation, which is given by the tax on income received from capital pensions, and further term divided according to personal income times an implicit tax rate:

$$T_{a,t}^{OPers} = T_{a,t}^{CapPension} + y_{a,t}^{Personal} \cdot \tau_t^{PRNCP}$$

[102] From data on the stock and from data on tax revenues we calculate the tax rate which we then forecast.

[103] A revaluation is a capital gain that is not realized (where assets change prices but are not traded). A capital gain occurs when the asset is traded.

[104] There is an abuse of notation relative to the labor market chapter where $n_{a,t}^e$ denotes only the employment of residents and not, as here, the employment of all workers aged a in period t.

[105] The last term is modifying the age dependent wage to be after civil servants contribution. This is modelled with the extra term as this contribution is not age dependent.

Tax on income received from capital pensions is given by

$$T_{a,t}^{CapPension} = \tau_t^{CapPension} \cdot f_t^{\tau CapPension} \cdot y_{a,t}^{CapPension}$$

The weight charge on cars, $T_{a,t}^{Weight}$, is calculated per person by an implicit tax rate times the stock of privately owned cars distributed by age according to non-housing ($\neg H$) consumption:[106]

$$T_{a,t}^{Weight} = \tau_t^{Weight} Cars_{t-1} \frac{C_{a,t}^{\neg H}}{C_t^{\neg H}} = \tau_t^{Weight} Cars_{t-1} \frac{N_{a,t} C_{a,t}^{\neg H}}{\sum_j N_{j,t} C_{j,t}^{\neg H}} \frac{1}{N_{a,t}}$$

Corporate taxation is given by:

$$T_t^{Corp} = \sum_{sp} T_{sp,t}^{Main} + T_{ext,t}^{NorthSea}$$

which is the corporate tax revenue from the private sector, T_t^{Main}, plus a tax for oil and gas extraction in the North Sea $T_t^{NorthSea}$. The corporate tax is levied on earnings before taxes, EBT_t :

$$T_t^{Main} = \tau_t^{Corp} \cdot \sum_{j \in sp \neg ext} EBT_{j,t}$$

while tax revenue from oil and gas extraction is given by:

$$T_{ext,t}^{Corp} = T_{ext,t}^{NorthSea} = \tau_t^{CorpNorth} \cdot EBITDA_{ext,t}$$

The taxation of the extraction sector is subject to the implicit tax rate $\tau_t^{CorpNorth}$, and based on earnings before taxes, interests and depreciations in the sector, $EBITDA_{ext,t}$.

Pension funds pay tax on the return to their financial assets (interest on bonds, dividends and capital gains on stocks).

$$T_t^{PAL} = \tau_t^{PAL} \cdot r_t^{return} \cdot A_{t-1}^{PFunds}$$

Finally, contributions to the public media are given by a fixed amount payed by each adult, τ_t^{Media}, multiplied by the number of adults.

$$T_t^{Media} = \tau_t^{Media} \sum_{a \geq 18} N_{a,t}$$

7.1.2 Income terms and allowances

Personal income is given by:

$$y_{a,t}^{Personal} = \left(w_{a,t} \frac{n_{a,t}^e}{N_{a,t}} - T_{a,t}^{AM} + TR_{a,t}^{Taxable} - PP_{a,t}^X - PP_{a,t}^{Cap} + y_{a,t}^{PX} \right) \cdot J_{a,t}^{yPersonal}$$

which is wage income per person excluding the labor market contribution, $T_{a,t}^{AM}$, plus taxable income transfers, $TR_{a,t}^{Taxable}$, defined below under government expenses, minus tax deductible pension payments to the two different types of pension systems, $PP_{a,t}^X$ and

[106] To make the model more consistent we could have age specific car stocks. Then the distribution of weight tax on age could be consistent to the prior car consumption by age. As we do not have car consumption divided by age we assume it to be proportional to overall consumption, thus sparing the extra book keeping by age, and distribute the tax according to non-housing consumption.

$PP^{Cap}_{a,t}$, plus taxable pension (received) income $y^{PX}_{a,t}$.[107] Pensions are discussed in the household chapter. An adjustment factor, $J^{yPersonal}_{a,t}$, ensures that the personal income matches imputed data.

Taxable income adds net capital income and subtracts a number of allowances (AL) defined below:

$$y^{Tax}_{a,t} = \left(y^{Personal}_{a,t} + y^{NetCapital}_{a,t} - AL^{EITC}_{a,t} - AL^{Unemp}_{a,t} - AL^{EarlyRet}_{a,t} - AL^{Other}_{a,t}\right) \cdot J^{yTax}_{a,t}$$

Net Capital Income of an average person in a given cohort is the difference

$$y^{NetCapital}_{a,t} = y^{Cap^{+}}_{a,t} - y^{Cap^{-}}_{a,t}$$

where positive capital income is the return on household nominal deposits and bonds $r^{dep} \cdot v^{HHdep}_{a,t}$ and $r^{bonds} \cdot v^{HHBonds}_{a,t}$ and negative capital income consists of interest payments on nominal bank debt $r^{debt} \cdot v^{HHBankDebt}_{a,t}$, and mortgage debt $r^{mort} \cdot v^{HHMort}_{a,t}$.[108]

All capital income is then part of taxable income and so enters the tax base for municipal taxation. However, only positive net capital income (above a certain threshold, $y^{NetCapital}_{a,t} > \underline{y} > 0$) is part of the tax base for bottom and top taxation. We are looking at micro data for an accurate measure and until then we include in the tax base for municipal taxation the following quantity:

$$y^{NetCap^{+}}_{a,t} = \left(y^{NetCapital}_{a,t} > \underline{y} > 0\right) \equiv y^{Cap^{+}}_{a,t} \cdot 0.5$$

The potential personal allowance is the same for every (adult) person and follows the indexation of transfers (satsregulering, s^{reg}). The actual average personal allowance used is, however, not the same for all cohorts as some (few) persons do not have an income[109]:

$$AL^{Pers}_{a,t} = AL^{Pers}_{a,t-1} \cdot s^{reg}_{t} + J^{ALPers}_{a,t}$$

The earned income tax credit (Beskæftigelsesfradrag, EITC) is an allowance for people in employment. It is a percentage of income up until a limit. It has the properties of a negative marginal tax for people with low income and a negative lump sum tax for people with high income. It is treated as a negative marginal tax, but with a tax rate equaling the average relative allowance. It could be distributed on age groups according to register data, but in this model version it is assumed to be the same for all age groups. This means the total tax credit can be calculated as the average allowance rate times wages:

$$AL^{EITC}_{t} = \tau^{EITC}_{t} \cdot w_t$$

The total tax credit is divided between the age groups of the population according to their share of wages:

$$AL^{EITC}_{a,t} = \left[\frac{w_{a,t} \cdot n^{e}_{a,t}}{\sum_a w_{a,t} n^{e}_{a,t}}\right] \cdot \frac{AL^{EITC}_{t}}{n_{a,t}}$$

[107]Income received from capital pensions is not taxed as personal income, but with an independent tax rate. Payments into capital pension are tax deductible.

[108]Capital income fits macro data from statistikbanken.dk and age profiles from registerdata.

[109]The personal allowance can be used by a spouse if a person has no income (and is married). This effect is not captured in the model.

Allowance for contribution to unemployment insurance, $AL^{Unemp}_{a,t}$, and allowance for contribution to early retirement (Efterløn), $AL^{EarlyRet}_{a,t}$, follow the contributions[110]:

$$AL^{Unemp}_{a,t} = A2C^{Unemp}_t \cdot Cont^{Unemp}_{a,t}$$

$$AL^{EarlyRet}_{a,t} = A2C^{EarlyRet}_t \cdot Cont^{EarlyRet}_{a,t}$$

We have data for $AL^{Unemp}_{aTot,t}$ and $Cont^{Unemp}_{aTot,t}$. The age decomposition follows wage income.

Other allowances include allowances for transport, clothes etc. These are primarily related to employment and therefore modeled to follow it:

$$AL^{Other}_t = ALR^{Other}_t \cdot n^e_t$$

and it is distributed among age groups according to hours worked:

$$AL^{Other}_{a,t} = \frac{n_{a,t}h_{a,t}}{\sum_a n_{a,t}h_{a,t}} \cdot \frac{AL^{Other}_t}{n_{a,t}}$$

7.1.3 Indirect taxation

Indirect taxes consist of value added taxes, excise duties, duties from car sales (a registration tax, registreringsafgifter), and production taxes. Value added taxes and the different duties are described elsewhere. Indirect taxes also include the difference between customs taxes (taxes on imported goods) and indirect taxes to the EU.

$$T^{Indirect}_t = T^{VAT}_t + T^{EDuty}_t + T^{Reg}_t + T^{Production}_t + T^{Cus}_t - T^{EU}_t$$

Indirect taxes to the EU is not exactly equal to customs so a correction factor is added:[111]

$$T^{EU}_t = f^{TCus}_t \cdot T^{Cus}_t$$

Revenues from most indirect taxes are coded in the taxes.gms file and explained in the chapter covering the input-output system. *Product* taxes described in the input output chapter are the main part of indirect taxes. They include VAT, customs taxes and duties.

There are, however, also *production* taxes. They consist of weight charges on cars, payroll taxes, taxes related to firm's contribution to workers education, and a small sum of other production taxes. The first three taxes are sector specific. The respective tax revenues are modeled using sector specific tax rates times the value of building capital, machinery capital and the wage sum of employees. *Production* taxes also include property taxes related to land and these are also sector specific.[112]

7.1.4 Other government revenues

The specific modeling of other revenues is not yet complete. Currently they are as follows:

$$\begin{aligned} T^{Other}_t = \quad & T^{Bequest}_t + T^{Church}_t + T^{\delta}_t + Cont_t \\ & + Rev^{Foreign}_t + Rev^{HHFirms}_t + \Pi^G_t \\ & + G^{LRent}_t + J^{GovRev}_t \end{aligned}$$

[110] Allowances include contribution and administration cost. Therefore the ratios of allowance to contribution $A2C^{Unemp}_t$ and $A2C^{EarlyRet}_t$ can be above 1.

[111] We have almost excatly $T^{Cus}_t = T^{EU}_t$.

[112] In the national accounts this land tax is paid by both firms and households. As firms do not own land in MAKRO the revenue is based on the capital stock of buildings and houses.

Bequest taxes (kapitalskatter/arveafgift) follow the bequest amount, $T_t^{Bequest} = \tau_t^{Bequest} \cdot Beq_t$, where $\tau_t^{Bequest}$ is the implict tax rate and Beq_t is the sum of bequests described in the household chapter. Tax revenue from the church tax follows the same tax base as municipal taxation and is (at a personal level) given by:

$$T_{a,t}^{Church} = \tau_t^{Church} \cdot f_t^{\tau Church} \cdot \left[y_{a,t}^{Taxable} - y_{a,t}^{PA}\right]$$

we leave the correction factor, $f_t^{\tau Church}$, explicit in the text because it also captures the fact that the church tax is not mandatory and therefore not all people pay it.

Revenues from the depreciation of government capital T_t^{δ} are discussed in the chapter on government production and consist of depreciation allowances paid by the government to itself. They are included here as revenue, while on the expenditure side they are a part of government consumption. On the public production side depreciation is counted as a cost. The public sector gets the money back here, however so the actual capital expense is the investment.
Contributions to social programs (Bidrag til social ordninger) $Cont_t$, sums a list of different specific payments to the state:

$$\begin{aligned} Cont_t = \quad & Cont_t^{Unemp} + Cont_t^{EarlyRet} + Cont_t^{FreeRest} \\ & + Cont_t^{Mandatory} + Cont_t^{CivilServants} \end{aligned}$$

All these contributions follow the labor force through a relation of the form:[113]

$$Cont_t^X = \mu_t^X \cdot n_t^{LabForce}$$

with contribution rate μ_t^X.[114] The set X contains contributions to early retirement, other voluntary contributions ($FreeRest$ = øvrige frivillig bidrag), mandatory contributions (obligatoriske bidrag), and contributions to civil servants pensions (bidrag til Tjenestemandspension).

Payments from foreign countries $Rev_t^{Foreign}$, payments from households and domestic firms $Rev_t^{HHFirms}$, and profits from public corporations Π_t^G, are all calibrated to match their respective shares of GDP. For example, given GDP and the share of government profit in GDP $\alpha_t^{\Pi G}$, the quantity Π_t^G is calculated as:

$$\Pi_t^G = \alpha_t^{\Pi G} \cdot GDP_t$$

We have data for Π_t^G and in the forecast period $\alpha_t^{\Pi G}$ is exogenously forecast using ARIMA. When we shock the model Π_t^G is exogenous.

The land rent, G_t^{LRent}, depends on the gross value added in the extraction sector, and is given by:

$$G_t^{LRent} = \tau_t^{Rent} \cdot GVA_{ext,t}$$

Lastly, J_t^{GovRev} is a variable that secures that $REV_t = T_t$ fits actual data. J_t^{GovRev} is a very small amount which fluctuates around zero and is set to zero in the forecast.

[113] The labor force in MAKRO is calculated exogenously (mechanically) as a function of endogenous employment as exogenous population.
[114] This rate follows the regulation rate of public transfers (sats-regulering) as explained under public expenditures.

Table 7.1: Government Revenues: Tax Rates.

Direct Taxes		
Text	Code	Factor Value
$\tau_{a,t}^{Bot}$	$= t_t^{Bot} \times f_{a,t}^{tBot}$	$f_{a,t}^{tBot} \neq 1$
τ_t^{Top}	$= t_t^{Top}$	
$\tau_{a,t}^{Municipal}$	$= t_t^{Municipal} \times f_{a,t}^{tMunicipal}$	$f_{a,t}^{tMunicipal} \neq 1$
$\tau_t^{Property}$	$= t_t^{Property}$	
τ_t^{Stocks}	$= t_t^{Stocks}$	
$\tau_t^{Business}$	$= t_t^{Business}$	
τ_t^{Death}	$= t_t^{Death}$	
τ_t^{AM}	$= t_t^{AM} \times f_t^{tAM}$	$f_t^{tAM} \neq 1$
$\tau_{a,t}^{PAL}$	$= t_t^{PAL} \times f_{a,t}^{tPAL}$	$f_{a,t}^{tPAL} \neq 1$
$\tau_t^{Bequest}$	$= t_t^{Bequest}$	
τ_t^{rent}	$= t_t^{Rent}$	
τ_t^{Church}	$= t_t^{Church} \times f_t^{tChurch}$	$f_t^{tChurch} \neq 1$
τ_t^{Weight}	$= t_t^{weight}$	
$\tau_t^{CapPension}$	$= t_t^{CapPension} \times f_t^{tCapPension}$	$f_t^{tCapPension} \neq 1$
τ_t^{Corp}	$= t_t^{Corp} \times f_t^{tCorp}$	$f_t^{tCorp} \neq 1$
τ_t^{Media}	$= t_t^{Media}$	

7.2 Expenditures

Government expenditures are given by consumption, investment, transfers, subsidies, and other expenses:

$$G_t^{EXP} = G_t^{Cons} + G_t^{Inv} + G_t^{Trans} + G_t^{Subs} + Other_t$$

Details of government investment and capital stock are described in the chapter on public production. Due to specific accounting standards which apply to the public sector, government consumption consists of two separate objects, one given by capital depreciation and the other being a quantity which follows population changes (with a population metric F_t^N) and wages. As detailed in the public production chapter we have for consumption excluding depreciation

$$P_t^{GCxD} G_t^{CxD} = \mu_t^{GCxD} F_t^N W_t$$

7.2.1 Transfers

Government income transfers are the sum of many different items (33 items j in the set Γ):

$$TR_t = \sum_{j \in \Gamma} TR_{j,t}$$

Every income transfer j is determined as a rate per person in million kr times a base in thousand persons:[115]

$$TR_{j \in \Gamma, t} = Rate_{j \in \Gamma, t} Base_{j \in \Gamma, t} + J_{j \in \Gamma, t}^{TR}$$

Rates follow the sats-regulering (SREG) rate:

$$Rate_{j \in \Gamma, t} = Rate_{j \in \Gamma, t-1} SREG_t + J_{j \in \Gamma, t}^{Rate}$$

The "sats-regulering" rate is based on the average wage per worker with a two year lag:

$$SREG_t = \frac{\frac{1}{n_{t-2}^e} \sum_a \left(w_{a,t-2} \cdot n_{a,t-2}^e \right)}{\frac{1}{n_{t-3}^e} \sum_a \left(w_{a,t-3} \cdot n_{a,t-3}^e \right)} + J_t^{SReg}$$

while the base is a mapping S2T from socio-economic groups contained in the demographic projection, "Befolkningsregnskabet" to the transfer groups.[116]

$$Base_{j \in \Gamma, t} = Base_{j \in \Gamma} = \sum_{soc \in Socio} S2T_{j \in \Gamma, soc} N_{soc}$$

The mapping S2T is contained in a matrix. In most cases this matrix only has diagonal elements - i.e. one socio-economic group receives one type of transfer. In several cases, however, more than one socio-economic group receives the same transfer type for example employed and not employed student receiving student benefits. Also, the base for some transfers is all people of age 18 and above. In a few cases the socio-economic groups are divided between two transfer groups where they are not the only recipients. This makes it necessary to have coefficients less than one in some cells.

[115] It is necessary to include an adjustment term in order to calibrate the model as in some years transfers have been paid even though the base is zero. This is probably due to corrections in transfers paid from the year before. The numbers are, however, very small and in projections this adjustment term is set to zero and not used.

[116] S2T stands for $Socio2Transfer_{j,soc}$, where $j \in \Gamma, soc \in Socio$.

The number of persons in the different socio-economic groups changes with employment. When employment increases by 1.000 persons groups that make up employment increase by 1.000 persons, and the groups which make up non employment decrease by the same 1.000 persons. The specific allocation follows the deviation from structural employment:

$$n_{soc,t} = \tilde{n}_{soc,t} + \lambda_{soc,t}^{dS2dE} \left(n_t^E - \tilde{n}_t^E \right) + J_{soc,t}^{nsoc}$$

where $n_t^E$is total employment (excluding foreign workers) and $\tilde{n}_t^E$ is the equivalent structural employment, and the factor $\lambda_{soc,t}^{dS2dE}$ is the marginal effect from deviations of employment relative to its structural level on the composition of the different population groups (socio-economic, index *soc*). The object $\tilde{n}_{soc,t}$ is the structural number of persons in the socio economic group *soc*, and $J_{soc,t}^{nsoc}$ is a calibration adjustment term. The adjustment term is set to 0 going forward, but can be non-zero in historical data.

Changes in employment affect not only the distribution of the population into socio-economic groups, but also the size of different groups receiving government income transfers. [117] The different socio-economic and transfer groups are not age-specific in the model. We do, however, need to know the total value of transfers divided by age - and how they are divided into taxable and non-taxable transfers. The government income transfer per person of a given age consists of several terms: a term consisting of transfers moving with employment (employment effect EEFF), a term consisting of children related transfers, and a term consisting of transfers not moving with employment (other effect OEFF):[118]

$$TR_{a,t} = Rate_t^{EEFF} \frac{n_{a,t}^e}{N_{a,t}} + TR_{a,t}^{children} + Rate_{a,t}^{OEFF}$$

The changes in transfers per employed that move with employment (Employment Effect Rate) are calculated to be in accordance with the effect on base for transfers above:

$$Rate_t^{EEFF} = \sum_{j \in \Gamma} \left(Rate_j \sum_{soc \in Socio} S2T_{j,soc} \lambda_{soc,t}^{dS2dE} \right)$$

The rate concerning employment effects is not age dependent as it is assumed that movements in employment cause the same effect on socio-economic groups no matter the age distribution of the employment changes.[119] The rate concerning other effects (OEFF) is given by:

$$Rate_a^{OEFF} = F^{OEFF} \cdot F_a^{OEFF} \cdot \left(\sum_{j \in \Gamma} Rate_j \sum_{soc \in Socio} S2T_{j,soc} P2S_{soc} + J_j^{Rate} \right)$$

This component does not move with employment. It is age distributed in order to capture the detail that not all age groups are allocated identically across socio-economic groups. The term in brackets is the average transfer per person over all age groups. Historically the age distributed transfers are imputed using age distributed socio-economic groups from

[117] The effect is based on estimations from the Ministry of Finance reported in the paper "Tilpasning af undergab i befolkningsregnskabet".

[118] The first rate is not per employed as employment is distributed among different socio-economic groups depending on age. It is the marginal effect that is assumed to be the same across age groups. So the first term is not the actual transfer per employed, but the marginal transfer evaluated at actual employment. Differences between average and marginal rates are caught in the second term and assumed to be unaffected by changes in employment.

[119] This assumption may be loosened in a later model version given more detailed data work.

BFR under the assumption that all recipients of a certain group receive the same amount independent of age. In order to match this imputed data an age-dependent factor F_a^{OEFF} is included. It is calibrated to catch all differences in transfer rates not from employment. In the projection the age-dependent factor is "pre-calibrated" outside the model using the projection from BFR and the projection for transfers. This ensures that also in the projection the factor represents the correct age-distribution of socio-economic groups. The non age-dependent other factor F^{OEFF} is endogenous and proportionally scales the non-employment-related transfers so the total age-distributed transfers yield the correct amount.[120] The composition effect not captured by the exogenous age distributed factor is small and the non age distributed factor is approximately 1.

Not all government income transfers are subject to income tax. The set of non taxed transfers is a subset of Γ denoted by $(\Gamma\neg\tau)$. All income transfers related to groups affected by changes in employment are taxed. Therefore, changes in employment do not change the amount of non-taxed transfers. Again, using age distributed socio-economic groups from BFR under the assumption that all recipients of a certain group receive the same amount independent of age, we calculate how the non-taxed transfers are distributed across age-groups as follows:

$$TR_{j\in(\Gamma\neg\tau),a} = \frac{Rate_{j,a}TR_j}{N_a}$$

This rule only influences the age distribution. The rule is updated when the population data (BFR) is updated or when the underlying rates change. The age distributed transfers subject to taxation are the subset of transfers denoted by $(\Gamma\tau)$, such that $\Gamma = (\Gamma\neg\tau)+(\Gamma\tau)$.

7.2.2 Subsidies

Government subsidies are given by subsidies for products and production minus subsidies financed by the EU:

$$S_t^{Sub} = S_t^{Product} + S_t^{Production} - S_t^{EU}$$

Production subsidies are mostly related to input costs, mainly wages. Production subsidies excluding those related to labor/wages are modeled as a constant share of gross value added. Product subsidies are negative duties which are part of the net duty rate. Both types of subsidies are determined in the taxes.gms module. Subsidies financed by the EU are modeled as an exogenous share of GDP. Expenditures on the purchase of land and of licenses, payments to foreign countries, to households and to domestic firms are all modeled as shares of GDP.

7.3 Net interest income

Net interests income consists of earned interest income from government assets minus paid interest on government liabilities:

$$Netr_t^y = \sum_{i\in\mathcal{A}} A_{i,t-1}\cdot r_{i,t} - \sum_{j\in\mathcal{L}} L_{j,t-1}\cdot r_{j,t}$$

where $\mathcal{A}$ is the set of Assets owned by government and $\mathcal{L}$ the set of government liabilities. Government assets consist of bonds, deposits and (almost exclusively domestic) equity,

[120] Without this term this would not be the case outside the calibration as the age dependent factors are only an ad hoc representation of the correct mechanism when transfer rates and or socio groups change. This is the price to pay for not having the age dimension on all socio economic transfer groups and adding approximately one million extra equations and doubling the size of the entire MAKRO model.

while government liabilities consist only of bonds (divided between real estate bonds and other bonds). In the forecast the above expression is written in terms of average rates and total nominal assets and liabilities. In the case of assets these are held in tow separate accounts, A and F, which we discuss below. We have:

$$Netr_t^y = \left(A_{t-1}^{\mathcal{G}} + F_{t-1}^{\mathcal{G}}\right) r_t^{\mathcal{A}} - L_{t-1}^{\mathcal{G}} r_t^{\mathcal{L}}$$

$$r_t^{\mathcal{A}} = \sum_{i \in \mathcal{A}} \omega_{i,t-1}^{\mathcal{G}} \cdot r_{i,t} + J_t^{r\mathcal{A}}$$

$$r_t^{\mathcal{L}} = \sum_{j \in \mathcal{L}} \omega_{j,t-1}^{\mathcal{G}} \cdot r_{j,t} + J_t^{r\mathcal{L}}$$

with

$$\omega_{i,t-1}^{\mathcal{G}} = \frac{A_{i \in \mathcal{A},t-1}}{A_{t-1}^{\mathcal{G}} + F_{t-1}^{\mathcal{G}} \equiv \sum_{i \in \mathcal{A}} A_{i,t-1}}$$

$$\omega_{j,t-1}^{\mathcal{G}} = \frac{L_{j \in \mathcal{L},t-1}}{L_{t-1}^{\mathcal{G}} \equiv \sum_{j \in \mathcal{L}} L_{j,t-1}}$$

The adjustment terms ensure that we match the observed historical return. Since not all individual assets (stocks of a specific company) yield the same return, a different micro composition of public and private portfolios implies different observed returns in the data. In the model equity is treated as an homogeneous asset. In the projection it must therefore generate the same return to all agents holding it which means setting the j-terms to zero.[121]

It is assumed that the value of government assets is a given fraction of GDP, $A_{t-1}^{\mathcal{G}} = A2Y_t{\cdot}GDP_t$, which implies changes in the primary budget relative to GDP change also the gross debt to GDP ratio. The interest rate on liabilities (government bonds) is the rate used as the government discount rate in calculating the indicator for fiscal sustainability. Government liabilities are residually given after government assets and government net wealth has been determined:

$$A_t^{\mathcal{G}} - NETW_t^{\mathcal{G}} = L_t^{\mathcal{G}}$$

Since we obtain liabilities as the residual object, we need an independent way of calculating net wealth. Government net wealth is determined as net financial assets excluding those in government funds ($F_t^{\mathcal{G}}$):

$$NETW_t^{\mathcal{G}} = A_t^{\mathcal{G}} - L_t^{\mathcal{G}} = NFA_t^{\mathcal{G}} - F_t^{\mathcal{G}}$$

where we note that this net financial assets object is not the same as $NFA_t^{\mathcal{G}} \neq A_t^{\mathcal{G}} - L_t^{\mathcal{G}}$. Government funds are exogenous. These are public savings available to be disbursed to private agents, and which are (financially) managed by the public sector until they are paid out. From an accounting view they are indistinguishable from any other asset portfolio the government may hold.

Net financial assets change with the government budget and with revaluations.[122]

$$NFA_t^{\mathcal{G}} = NFA_{t-1}^{\mathcal{G}} + Budget_t + REV_t^{\mathcal{G}}$$

[121] It is assumed that the government does not issue mortgages or equity and only has debt in the form of bonds.

[122] When measuring the value of assets it is irrelevant whether assets are traded or not. Therefore a revaluation is the same as a capital gain. The distinction is only relevant for tax purposes.

Revaluations are modeled in the same way as dividends or interest payments - i.e. as a weighted average of revaluations for the different types of assets and liabilities with a j-term to adjust for historically different returns for the different sectors:

$$REV_t^{\mathcal{G}} = \sum_{i \in \mathcal{A}} A_{i,t-1} \cdot r_{i,t}^{cgain} - \sum_{j \in \mathcal{L}} L_{j,t-1} \cdot r_{j,t}^{cgain} + J_t^{\mathcal{G}rev}$$

The revaluation/capital gain rate for equity is the change in the value of the firm which is endogenous in the model. All other revaluation rates are exogenous. The different types of assets except bonds are a constant part of total assets including funds (constant portfolio weights):[123]

$$A_{i,t}|_{i \in \mathcal{A}} = \omega_{i,t}^{\mathcal{G}} \cdot \left(A_t^{\mathcal{G}} + F_t^{\mathcal{G}}\right)$$

Liabilities are also equally divided between bonds and mortgages:

$$MORG_t^{\mathcal{G}} = \omega_{morg,t}^{\mathcal{G}} \left[\sum_{j \in \mathcal{L}} L_{j,t}\right] = \omega_{morg,t}^{\mathcal{G}} L_t^{\mathcal{G}}$$

$$\omega_{morg,t}^{\mathcal{G}} = \frac{MORG_t^{\mathcal{G}}}{L_t^{\mathcal{G}}}$$

These two equations are identical. In data years the bottom expression is used to calculate omega, and in the forecast years an exogenous omega is used to calculate mortgages.

Non-mortgage Bonds are the net of assets and liabilities:

$$Bonds_t^{\mathcal{G}} = \omega_{Bonds,t}^{\mathcal{G}} \cdot \left(A_t^{\mathcal{G}} + Funds_t^{\mathcal{G}}\right) - \underbrace{\left(1 - \omega_{morg,t}^{\mathcal{G}}\right) \cdot L_t^{\mathcal{G}}}_{\text{Government-issued Non-mortgage Bonds}}$$

7.4 Structural objects

The government structural budget is given as the actual budget corrected for business cycle effects and other temporary effects:

$$SBdg_t = Bdg_t - BCEff_t - OTEff_t$$

The business cycle effect is calculated on the basis of a budget elasticity, the output-gap and the employment-gap[124]:

$$BCEff_t = \eta_t^{Budget} \cdot \left(0.6 \cdot \left(\frac{n_t^e}{\tilde{n}_t^e} - 1\right) + 0.4 \cdot \left(\frac{y_t}{\tilde{y}_t} - 1\right)\right) \cdot GVA_t$$

Other temporary effects consists of gaps in tax revenues (pension return tax, extraction tax, company taxation, registration duties on cars), gaps in net interest, gaps in other special posts and extraordinary corrections. These are in the current version of the model taken as exogenous.

The fiscal sustainability indicator, in the model called HBI (holdbarhedsindikator), is equal to the net present value of all future government revenues minus expenditures (primary budget) minus the initial government net debt (or plus net wealth) relative to the net present value of GDP:

[123] In the data, we only have a breakdown of assets and liabilities, where funds are included. We assume that funds have the same distribution as other public savings.

[124] Details on calculations of the structural budget balance is given in "Finansministeriets metode til beregning af strukturel saldo" available on the web page from the Ministry of Finance.

$$HBI_t = \frac{(1+r_t)\sum_{i=0}^{\infty}\left(\Pi_{j=0}^{i}\frac{1}{1+r_{t+j}}\right)PrBdg_{t+i} + NETW_{t-1}^{\mathcal{G}}}{(1+r_t)\sum_{i=0}^{\infty}\left(\Pi_{j=0}^{i}\frac{1}{1+r_{t+j}}\right)GDP_{t+i}}$$

where we discount both GDP and primary budgets using the government bonds rate. It is assumed that the primary budget balance and GDP is constant (corrected from underlying growth and inflation) from year 2099 and onward.

8 Public Production

The public sector output that is consumed in the economy consists of all the different goods and services provided by the state, from education and health care, to the judicial system and defense, and to child care, elderly care, etc. This output, Y_t^G, is not exported. It is entirely consumed domestically. The vast majority of these services are paid for using tax revenues or using the intake from public debt issues. A small amount is paid directly by private agents, as is the case for some co-payments for health care and education.

These private payments show up as household consumption of public goods and services, and we denote them here as C^{PU}. The rest is accounted for as government consumption of public goods and services, G^{PU}, or government investment from public production I^{PU}. This is a demand side view of the public output. In real quantities $Y_t^G = C^{PU} + \left(G^{PU} + I^{PU}\right)$.

Total demand for public goods equals total production of public goods. From the perspective of the supply side, the demand for public output is either exogenous or taken as given. In fact, in the model public consumption is partly exogenous as one of its key determinants is the exogenous evolution of population. In what follows we look at the supply side, namely at how Y^G is generated, and how respective prices are calculated. Afterwards we return to the demand side.

8.1 The Supply Side

There is one "supply function" for the entire public sector.[125] Y^G is produced just as private sector goods are, in the sense that it uses labor, capital equipment and structures, and intermediate inputs. Public production, however, differs from private production in three important details. First, in the data public production is measured by the input method. This means the value of output is exactly the sum of the value of the inputs into production. Second, following accounting standards for the public sector, the cost of public capital is entirely accounted for as depreciation. Investments into capital accumulation are not directly considered to be capital costs. These accounting rules imply we need alternative modeling to the production of public output. Third, there is no production function.

The input method is equivalent to a zero profit condition. We know that as long as we can measure the nominal cost of each input X_t^j we must have

$$P_t^0 Y_t^G = \sum_j P_t^j X_t^j$$

We have separate measures of the price and quantity of each input. We can measure investment, capital stocks, employment, and quantities of intermediate inputs used.

The remaining issue is how to measure separately the quantity of public output, Y^G, and its price P^0. In our model of the private sector we solve this problem using a theory of production. This is materialized in a (CES) production function that describes how the quantities of inputs are organized to generate units of output. The output price is then a by-product of this theory and of profit maximization. This is the optimization price P^0, the same derived here for the public sector.[126]

In the public sector we follow the data and use instead a "model" for the output price. This "model" is a price index. Given the zero profit condition the quantity of output can then be determined as the residual variable. There is no optimal choice of inputs as there is in the model of the private sector. Such a choice is replaced by rules for the evolution of

[125] The term supply function is used under caution since there is no production function of public goods.
[126] Details in the chapter on the problem of the firm, and the chapter on pricing.

input requirements which are taken as given by the government - if we view government as “managing” this giant production “firm”.

One of the assumptions regarding the evolution of inputs is that the capital stock follows the evolution of both public and private sector output. We make this assumption explicit below where we add the parameter $\hat{\alpha}_t^K$ to model it. Another assumption is that labor costs and intermediate input costs have an exogenous proportional relationship, here summarized by the parameter α_t^R.

For what follows, it is important to emphasize that the input method implies the equation above matches flows of inputs with an output flow. Flows of intermediate inputs and flows of labor costs are directly measured, and the remaining flow is that of costs related to capital. In a model of private sector production this flow would be closely related to investment, such as costs of investment and installation costs. Here, however, this flow is considered to be a measure of expenses with only capital depreciation.

8.1.1 Fundamentals

Input prices: materials and labor

The demand for intermediate inputs (materials) has the same structure as that for firms in the private sector. The price for materials in the public sector, p_t^R, is therefore determined just as in the private sector problem.

The measured expenditure on labor by the public sector consists of wages paid, $\hat{w}_t n_t$. Payroll taxes, τ_t^L, are disregarded here as they are a transfer from the state to itself. The wage expenditure also disregards vacancy posting costs as these are a component of the user cost of labor which is not considered in the input method of accounting for the public sector. The wages per worker in the public sector is then

$$\hat{w}_t = w_t \bar{h}_t \bar{\rho}_t \rho_t^g$$

where $\bar{h}_t$ and $\bar{\rho}_t$ are average hours and average worker productivity which are equal for all firms including the public sector, and ρ_t^g is a parameter that calibrates the different average wages across sectors. The unit wage w is the average contracted wage and reflects the wage rigidity due to staggered contract bargaining. All these are detailed in the labor market chapter. Below we work with the labor variable $L = h\rho n$, so that in this text these objects are relabeled $\hat{w}_t n_t = P_t^L L_t$.

Capital depreciation rates

Public capital stocks (machinery and buildings) each obey the standard law of motion

$$K_t = \left(1 - \delta_t^G\right) K_{t-1} + I_t$$

In the years where data are available we use observed investment and capital stock and apply the law of motion to obtain the depreciation rate δ_t^G.[127] This is important since the depreciation rate is a key parameter in the user cost of capital, and for public capital is is **the** key parameter. Then, given the historical data generated for the depreciation rate, we fit an ARIMA process to that data, and use it to forecast the future evolution of δ^G.

The mechanics of the law of motion are extended beyond the period with available data and into a planning horizon (2025) where we feed into the model the investment

[127]Every production sector has its specific building and materials capital depreciation rate since capital is the accumulation of a CES aggregation of investments sourced from all production sectors, and this sourcing varies across the demand side sectors.

expenditure planned by the government. This, coupled with the forecast of the depreciation rate, yields a time series for the capital stock for this "planning period". After the planning period the assumption of the relationship between the capital stock and public and private sector output embodied in the parameter $\hat{\alpha}_t^K$ is the constraint determining the evolution of investment. This is detailed below.

Input prices: capital

Following international accounting standards, in the national accounts depreciation alone is used as the cost of public capital. We denote this cost of public capital as proportional to the capital stock, $P_t^K K_{t-1}$, and P_t^K is given as the investment price of the relevant type of capital, P_t^I, times the depreciation rate calculated above, δ_t^G. In order to exactly match the data in the data periods we need two additional correction terms, λ, such that for each type of capital we create a new price variable $P_t^{I\lambda}$ and a new quantity variable K_t^λ as follows

$$P_t^K K_{t-1} \equiv P_t^{I\lambda} K_t^\lambda = \underbrace{\lambda_t^p P_t^I}_{P_t^{I\lambda}} \underbrace{\lambda_t^q \delta_t^G K_{t-1}}_{K_t^\lambda}$$

We have data measures for both $P_t^{I\lambda}$ and K_t^λ. Given our data on capital K_{t-1} and investment I_t we used the law of motion to recover the depreciation rate. Given the empirical measure of K_t^λ this then allows for the recovery of λ_t^q. Investment prices P_t^I and the empirical measure of $P_t^{I\lambda}$ allow for the identification of λ_t^p. The values of $(\lambda_t^q, \lambda_t^p)$ are very close to 1 in the data years so this is a small correction.[128] These are both eliminated (take the value 1) after 2017. We are therefore just valuing depreciation with the investment price, $P_t^I \delta_t^G K_{t-1}$.

We now detail how these new price and quantity variables are used in accordance with the way the output price index is constructed in the data.

8.1.2 Calculating the price of public production

Given our $P_t^{I\lambda}$ and K_t^λwe impose the labor-materials restriction. Define the residual value of labor-plus-materials by using the equation

$$V_t^{LR} = Y_t^G P_t^0 - \sum_{i \in (b,m)} P_{i,t}^{I\lambda} K_{i,t}^\lambda$$

Now add the assumption regarding the relationship between labor costs and intermediate input costs through the exogenous parameter α_t^R. This parameter is endogenous in the data years, and fixed/forecast after that. Define then expenditure on labor and materials by adding the equations

$$P_t^R R_t = \alpha_t^R V_t^{LR}$$

$$P_t^L L_t = \left(1 - \alpha_t^R\right) V_t^{LR}$$

With these, calculate the output price index as done in the data:

$$P_t^0 = P_{t-1}^0 \frac{\sum_{i \in (b,m)} P_{i,t}^{I\lambda} K_{i,t}^\lambda + P_t^R R_t + P_t^L L_t}{\sum_{i \in (b,m)} P_{i,t-1}^{I\lambda} K_{i,t}^\lambda + P_{t-1}^R R_t + P_{t-1}^L L_t}$$

[128]Total depreciation value $P_t^I \delta_t^G K_{t-1}$ differs slightly from the national accounts data due to compositional effects in the prices of capital and investment. This affects the chain indices used to calculate prices. In order to match the data exactly we need the λ factors.

and an initial condition for P_t^0 is required and also available in the data. The numerator on the right hand side equals by definition $P_t^0 Y_t^G$.

This equation, and in general the expressions in the algebra in this chapter look slightly different in the code, as there we have growth correction terms and other details which are not essential to the exposition here.

8.1.3 Determining real investment

Now we add the main restriction imposed by the exogenous number $\hat{\alpha}_t^K$:

$$P_t^I K_{t-1} = \hat{\alpha}_t^K \left(0.7 \times \left(P_t^{YG} Y_t^G - P_t^R R_t\right) + 0.3 \times X_t\right)$$

The auxiliary X_t is a measure of private sector value added. This cannot be analyzed as it is, because this capital stock K_{t-1} is already determined. So, the restriction that applies at time t is the above equation forwarded one period. Using the law of motion to eliminate K_t we obtain that time t investment is a forward looking quantity that solves only in the full model equilibrium:

$$I_t = \frac{\hat{\alpha}_{t+1}^K}{P_{t+1}^I} \left(0.7 \times \left(P_{t+1}^{YG} Y_{t+1}^G - P_{t+1}^R R_{t+1}\right) + 0.3 \times X_{t+1}\right) - \left(1 - \delta_t^G\right) K_{t-1}$$

The parameter $\hat{\alpha}_t^K$ is endogenous in the data years. It is implied by the available data on investment. This is reversed after the planning period where the exogenous forecast of $\hat{\alpha}^K$ implies investment.

We notice here the presence of a new price variable, P_t^{YG}. This variable differs from P_t^0 in the data years but it is virtually identical after 2016 (a nearly constant factor difference of 2‰).

8.1.4 Matching the code

We have a large number of objects. They are labeled in the code as shown in Table 1.

8.2 Composition and determination of investment and intermediate inputs

As mentioned at the start of this chapter, the evolution of the size of government is partially exogenous. Not only that, some specific components of government expenditure also follow exogenous trends or predetermined relationships to aggregate variables.

8.2.1 Intermediate inputs

The objects P_t^R = pR['off',t] and R_t = qR['off',t] are aggregates of purchases by the government from all sectors in the economy, and also from abroad, just as in the private sector. Just as detailed in the consumption chapter, the quantity R_t is sourced first from all nine production sectors using a Leontief structure. In terms of parameters we have only the fixed proportion (scale) parameters. The government obtains intermediate inputs mostly from manufacturing and services with smaller but significant contributions from energy and construction. We have

$$R_t = min\left(\frac{R_t^{man}}{\mu^{man}}, \frac{R_t^{ser}}{\mu^{ser}}, \text{etc,}\right)$$

or equivalently $R_t^{man} = \mu^{man} R_t$, and $R_t^{ser} = \mu^{ser} R_t$, etc, with $\sum_j \mu^j = 1$. In 2017 these parameters have the values shown in Table 2.

In the code these parameters are labeled μ^s = uIO['off',s,t]. This indexing merits explanation. The code object uIO[x,s,t] maps the demand set x against the supply set of nine production sectors s. In the case of intermediate inputs the set is $x = r$ and maps s into s because the general construction is that all nine sectors purchase inputs from each other.

Below that, the sourcing from foreign and domestic suppliers is done through CES aggregation.

8.2.2 Investment

In the data we have different classifications of investment which have to be allocated to our two types of capital goods. These are direct, indirect, and new investments. Indirect investments are purchases of existing capital and are entirely allocated to structures (buildings) capital. New investments are divided between both capital types with a share parameter, $\mu_{b,t}^{NEW}$. And direct investments, which consist almost entirely of publicly funded R&D are allocated to machinery investment.

Define a value object as equal to a price measure times a quantity measure. For any index A we have that the nominal value of some type A of investment is given by $V^A = p^A I^A$. Public investment, V_t^{GI}, then consists of direct investment V_t^{DIR}, indirect investment, V_t^{IND}, and new investment, V_t^{NEW}. They map into buildings and machinery as follows:

$$V_{b,t}^{GI} = \mu_{b,t}^{NEW} V_t^{NEW} + V_t^{IND}$$

$$V_{m,t}^{GI} = \left(1 - \mu_{b,t}^{NEW}\right) V_t^{NEW} + V_t^{DIR}$$

Total investment is then

$$V_t^{GI} = V_{b,t}^{GI} + V_{m,t}^{GI} = V_t^{NEW} + V_t^{IND} + V_t^{DIR}$$

The values of public direct and indirect investments are given by fixed factors, μ_t^{DIR} and μ_t^{IND}, times nominal value added in the economy

$$V_t^{DIR} = \mu_t^{DIR} V_t^{BVT}$$

$$V_t^{IND} = \mu_t^{IND} V_t^{BVT}$$

Finally, the price of direct investment is the price of public output since the state is effectively purchasing the goods that it is producing. The price of indirect investment is the price of structures (buildings), $P_{b,g,t}^{I}$. It has a sectoral index $g \equiv$ 'off' because the general construction is that investments are aggregates/compositions of purchases from all sectors and this composition can vary across demand-side sectors. In fact they do not as we impose the same sourcing structure across all sectors.[129] The price of new investments is an average of the sector specific prices of buildings and machinery

$$P_t^{NEW} = \mu_t^{PNEW} \left(\mu_{b,t}^{NEW} P_{b,g,t}^{I} + \left(1 - \mu_{b,t}^{NEW}\right) P_{m,g,t}^{I} \right)$$

and of course $V_t^{NEW} = P_t^{NEW} I_t^{NEW}$.[130]

In addition, the above relationships are used to impose exogenous structure on the data, not to calculate investment prices specific to the public sector. The reason is that these prices are assumed to be the same as in all other production sectors. The investment price of machinery and buildings is identical across all sectors because it is assumed to be sourced with the same composition in all sectors from all sectors, and also with the same domestic and foreign goods composition.

The parameters μ_t^{DIR}, μ_t^{IND}, $\mu_{b,t}^{NEW}$ are calibrated in order for V_t^{DIR}, V_t^{IND}, V_t^{NEW} to fit the available data.

8.2.3 In the code

Once again it is useful to translate these objects into code language and Table 3 contains a useful summary.

8.3 The demand side

Public production, Y_t^G, is given in the Input/Output system as the sum of three demand components: private consumption (of public services) C^{PU}, public consumption (of public services) G^{PU}, and public direct investments, I_t^{DIR}.

$$Y_t^G = C_t^{PU} + \left(G_t^{PU} + I_t^{DIR}\right)$$

In the planning horizon the nominal value of private consumption of public services $P_t^G C_t^{PU}$, and the nominal value of public direct investments, V_t^{DIR}, are both exogenized and together with the public price index they determine the quantities C_t^{PU} and I_t^{DIR}. The remaining demand side component is public consumption of public output, G_t^{PU}.

Total public consumption G_t is the sum of public consumption of public output plus public consumption of private output, $G_t = G_t^P + G_t^{PU}$. Both components are described in the IO-chapter. The nominal value of total public consumption, V_t^{GC}, is now further

[129] We actually add a very small correction factor because we do need it to vary across sectros in order for investment to exactly match the data.

[130] Direct investment is a particular item because conceptually it is an investment the public sector purchases from itself and yet it is priced at the price of machinery. We never actually use the quantity I_m^{DIR}, only its value V^{DIR}. The corresponding quantity could be recovered with the price $P_{m,t}^{I}$. However, only the total quantity of public investment into machinery is needed in order to use the law of motion for capital.

decomposed into two parts. One is the depreciation cost of capital which we have detailed in the supply side, and the other is the remaining amount which is "modeled" as evolving according to population with a factor F_t^N and total wage income in the economy, denoted here by W_t, and overall with a parameter μ_t^{GCxD}:

$$V_t^{GC} = \sum_{k\in(b,m)} P_{k,t}^{I\lambda} K_{k,t}^{\lambda} + \mu_t^{GCxD} F_t^N W_t$$

The associated quantity object $\left(\mu_t^{GCxD} F_t^N W_t\right) / P_t^{GCxD}$ is the real (as opposed to nominal) public consumption excluding the depreciation cost of capital, G_t^{CxD}. This quantity is a calibration object and it is exogenous in the planning period. This then requires the calculation of the specific price for such quantity, P_t^{GCxD}, and this calculation is done using a chain index as follows:

$$\mu_t^{GCxD} F_t^N W_t \equiv P_t^{GCxD} G_t^{CxD}$$

$$P_t^{GCxD} G_t^{CxD} = P_t^G G_t - \sum_{k\in(b,m)} P_{k,t}^{I\lambda} K_{k,t}^{\lambda}$$

$$P_{t-1}^{GCxD} G_t^{CxD} = P_{t-1}^G G_t - \sum_{k\in(b,m)} P_{k,t-1}^{I\lambda} K_{k,t}^{\lambda}$$

These equations contain the price of total public consumption, P_t^G, which is the composite of private sector prices and the price of public output. This is not the same object as the prices we saw above, P_t^{YG} and P_t^0. Further details of the construction of all prices can be found in the government expenditure chapter.

In the planning period real public consumption, G_t , is an endogenous variable determined in part by the exogenized G_t^{CxD}. Public production to public consumption, G_t^{PU}, is a fixed share of real public consumption, G_t. After the planning horizon G_t^{CxD} is endogenous and G_t is exogenized and set to follow a demographic development.

8.4 Appendices - Public Production

8.4.1 Productivity Growth

It is assumed the there is no labor augmenting technological progress in the public sector. A simple way to understand the consequences of this fact is to work as if public production happened through a Cobb-Douglas production function with inputs (K_b, K_m, L, R) . In such a case the price would be the variable recovered through the zero profit condition, and this price would be

$$P_t = \left(\frac{P_{b,t}^K}{\alpha_b^K}\right)^{\alpha_b^K} \left(\frac{P_{m,t}^K}{\alpha_m^K}\right)^{\alpha_m^K} \left(\frac{P_t^L}{\alpha^L}\right)^{\alpha^L} \left(\frac{P_t^R}{\alpha^R}\right)^{\alpha^R}$$

Consider now the effect of labor augmenting technological progress inside the production function. This generates the following price relationship

$$P_t = \left(\frac{P_{b,t}^K}{\alpha_b^K}\right)^{\alpha_b^K} \left(\frac{P_{m,t}^K}{\alpha_m^K}\right)^{\alpha_m^K} \left(\frac{P_t^L}{\alpha^L \xi_L}\right)^{\alpha^L} \left(\frac{P_t^R}{\alpha^R}\right)^{\alpha^R}$$

On a balanced growth path all input prices or user costs grow with the inflation rate, except for the user cost of labor which increases with the inflation rate plus the Harrod neutral growth rate g_ξ. This implies for a Cobb-Douglas output price:

$$\frac{P_t}{P_{t-1}} = (1+\pi)^{\alpha_b^K + \alpha_m^K + \alpha^R} \left(\frac{(1+\pi)(1+g_\xi)}{(1+g_\xi)} \right)^{\alpha^L} = 1+\pi$$

It is, however, assumed that there is no productivity growth in the public sector. This implies that its price will grow with a higher rate, namely

$$\frac{P_t^0}{P_{t-1}^0} = (1+\pi)^{\alpha_b^K + \alpha_m^K + \alpha^R} \left(\frac{(1+\pi)(1+g_\xi)}{1} \right)^{\alpha^l} = (1+\pi)(1+g_\xi)^{\alpha^L}$$

where the contribution of the growth rate of technology on the labor price is weighed by the labor share.

This is captured in the price index of public production by adding the growth of technology in the denominator as follows

$$P_t^0 = P_{t-1}^0 \frac{\sum_{i \in (b,m)} P_{i,t}^{I\lambda} K_{i,t}^{\lambda} + P_t^R R_t + \left(P_t^L / \cancel{\xi_t^L} \right) \left(\cancel{\xi_t^L} L_t \right)}{\sum_{i \in (b,m)} P_{i,t-1}^{I\lambda} K_{i,t}^{\lambda} + P_{t-1}^R R_t + \left(P_{t-1}^L / \xi_{t-1}^L \right) \left(\xi_t^L L_t \right)}$$

and this works in the desired way because the technology factor is normalized to be a constant equal to 1 for all sectors except for the public sector where it declines in value over time. It takes the value 1 in 2010 and then declines steadily (it reaches 0.5 between 2079 and 2080).

Table 8.1: Public Production Code Names. Part 1.

$P^{I\lambda}_{i,t}$	= pOffAfskr[k,t]	$\hat{\alpha}^{K}_{i,t}$	= rOffK2Y[k,t]	$K^{\lambda}_{i,t}$	= qOffAfskr[k,t]
P^{R}_{t}	= pR['off',t]	α^{R}_{t}	= rvOffR2LR[t]	R_t	= qR['off',t]
P^{L}_{t}	= vhW[t]	V^{LR}_{t}	= vOffLR[t]	L_t	= L['off',t]
P^{0}_{t}	= pKLBR['off',t]	$\lambda^{p}_{i,t}$	= fpOffAfskr[k,t]	Y^{G}_{t}	= qY['off',t]
		$\lambda^{q}_{i,t}$	= fqOffAfskr[k,t]	$I_{i,t}$	= qI_s[k,'off',t]
$P^{I}_{i,t}$	= pI_s[k,'off',t]	$\delta^{G}_{i,t}$	= rAfskr[k,'off',t]	$K_{i,t}$	= qK[k,'off',t]

'off' is an element of set 's' denoting the public sector. 'iB' is an element of the set 'k' denoting buildings, and 'im' denotes machinery in the same set.

Table 8.2: Intermediate input parameter values

μ^{man}	= 0.1802	μ^{con}	= 0.0430	μ^{hou}	
μ^{ser}	= 0.7155	μ^{ene}	= 0.0559	μ^{sea}	= 0.0036
μ^{agr}	= 0.0014	μ^{ext}	= 0.0004	μ^{gov}	

In the code these parameters are labelled μ^{s}=uIO['off',s,t]

Table 8.3: Public Production Code Names. Part 2.

$V^{GI}_{b,t}$	= vI_s['iB','off',t]	$\mu^{NEW}_{b,t}$	= rOffNyIB2I[t]
$V^{GI}_{m,t}$	= vI_s['im','off',t]	μ^{DIR}_{t}	= rvOffDirInv2BVT[t]
$P^{I}_{m,g,t}$	= pI_s['im','off',t]	μ^{IND}_{t}	= rvOffIndirInv2vBVT[t]
$P^{I}_{b,g,t}$	= pI_s['iB','off',t]	V^{NEW}_{t}	= vOffNYInv[t]
μ^{PNEW}_{t}	= fpOffNyInv[t]	V^{DIR}_{t}	= vOffDirInv[t]
P^{NEW}_{t}	= pOffNyInv[t]	V^{IND}_{t}	= vOffIndirInv[t]

Table 8.4: Public Production Code Names. Part 3

G_t^C	= qG['gtot't]	V_t^{GC}	= vG['gtot't]
μ_t^{GCxD}	uvGxAfskr[t]	F_t^N	= fDemoTraek[t]
W_t	= vhW[t]	P_t^{GCxD}	= pGxAfskr[t]
G_t^{GCxD}	= qGxAfskr[t]	P_t^{GC}	pG['gTot',t]

Text	Code	Definition
$P_{i,t}^{I\lambda}$	poffAfskr[k,t]	Deflator for public depreciation
P_t^R	pR["off',t]	Input deflator for materials
$P_t^L = W_t$	vhW[t]	Wage per unit of productive labour
$P_t^G = P_t^{GC}$	pG[gTot,t]	public consumption deflator
$P_{i,t}^K$	pK[k,'off',t]	User cost of public capital
$P_{i,t}^I$	pI_s[k,'off',t]	Investment deflator
$\hat{\alpha}_{i,t}^K$	rOffK2Y[k,t]	Public capital policy ratio
α_t^R	rvOffR2lR[t]	Share of materials expenditure
V_t^{LR}	vOffLR[t]	Expenditure on materials and labor
$\lambda_{i,t}^p$	fpOffAfskr[k,t]	Correction term for Pk
$\lambda_{i,t}^q$	fqOffAfskr[k,t]	Correction term for K
$\delta_{i,t}^G$	rAfskr[k,'off',t]	Capital depreciation rate
$K_{i,t}^\lambda$	qOffAfskr[k,t]	Total capital depreciation
R_t	qR['off',t]	Quantity on materials
L_t	L['off',t]	Total productive hours
Y_t^G	qY['off',t]	Public production quantity
$I_{i,t}$	qI_s[k,'off',t]	Public investment quantity
$K_{i,t}$	qK[k,'off',t]	public capital quantity
$V_{b,t}^{GI}$	vI_s['iB', 'off',t]	Value of Structures (buildings)
$P_{m,g,t}^I$	pI_s['im','off',t]	Investment deflator for machinery
μ_t^{PNEW}	fpOffNyInv[t]	factor
P_t^{NEW}	pOffNyInv[t]	Deflator for new investments
$\mu_{b,t}^{NEW}$	rOffNyIB2I[t]	Building capital's share of total public capital
μ_t^{DIR}	rvOffDirINv2BVT[t]	Direct investment to GVA
μ_t^{IND}	rvOffIndirINv2vBVT[t]	Indirect investment to GVA
V_t^{NEW}	vOffNYInv[t]	Value of new investments
V_t^{DIR}	vOffDirInv[t]	Value of direct investment
V_t^{IND}	vOffIndirInv[t]	Public sector net purchase of existing capital
G_t^C	qG['gTot',t]	Quantity of public consumption
μ_t^{GCxD}	uvGxAfskr[t]	Scale parameter
G_t^{GCxD}	qGxAfskr[t]	Public consumption excluding depreciation
V_t^{GC}	vG['gtot',t]	Value of public consumption
F_t^N	fDemoTraek[t]	Population factor
P_t^{GCxD}	pGxAfskr[t]	Deflator

9 Structural employment and structural GVA

Gaps in employment and gross value added (GVA) are inputs to the calculation of the structural budget balance in the Ministry of Finance. They are defined as the difference between actual and structural quantities. Structural employment is akin to steady state employment. Structural employment in MAKRO is calculated using a long run simplified version of the labor market from the actual model. Structural GVA is calculated from data, using structural employment, with the method described at the end of this chapter.

MAKRO also includes a structural labor force. Neither the labor force nor unemployment in MAKRO are determined endogenously. Instead, the model determines employment, and those extra variables are calculated from it using an exogenous rule. Therefore neither the structural labor force nor structural unemployment can affect structural employment or structural GVA.

9.1 Structural employment

According to the calculation principles of the Ministry of Finance, structural employment is to be regarded as steady-state employment. Employment levels per se cannot be constant as population changes even in the long run, but rate measures such as employment relative to population and unemployment rates can.

The labor market in MAKRO is constructed so that long run employment is independent of wages and prices. Here we provide a simplified model for calculating structural employment.

In the actual model, age specific employment has the law of motion:

$$n^e_{a,t} = \left[\underbrace{(1-\delta_a)\frac{N_{a,t}}{N_{a-1,t-1}}}_{1-\hat{\delta}_{a,t}}\right] n^e_{a-1,t-1} + x_t \cdot n^s_{a,t}$$

with

$$(1-\delta^n_t) = \frac{\sum_a \left(1-\hat{\delta}_{a,t}\right) n^e_{a-1,t-1}}{n^e_{t-1}}$$

so that

$$n^e_t = (1-\delta^n_t)\, n^e_{t-1} + \hat{x}_t n^s_t$$

where $n^e_{a,t} = N_{a,t} q^e_{a,t}$ is employment, δ_a is the job separation rate, x_t is the finding rate and $n^s_{a,t}$ is cohort total search effort, $n^s_{a,t} = N_{a,t} q^s_{a,t}$. The steady state counterpart to this aggregate law of motion is

$$n^{e*}_t = (1-\delta^{n*}_t)\, n^{e*}_{t-1} + x^*_t n^{s*}_t$$

where there is both a structural finding rate, x^*_t, and a structural search quantity, n^{s*}_t, both endogenous. The structural model uses mainly the aggregate law of motion with cohort specific objects as auxiliary quantities.

The firm side of the labor market is summarized by an aggregated version of the first order condition for vacancy posting which incorporates the Nash bargaining solution.

From the sector j expression in the model

$$1-\frac{\partial\left(\chi_{j,t}n_{j,t}\right)}{\partial n_{j,t}}=\frac{\hat{w}_{j,t}}{p^L_{j,t}\xi_{j,t}}+D^n_{j,t+1}\frac{p^L_{j,t+1}}{p^L_{j,t}}\left(\frac{\partial\left(\chi_{j,t+1}n_{j,t+1}\right)}{\partial n_{j,t}}\right)$$

we generate the aggregated object, using the Nash solution $w=\left(1-\phi^{\text{nash}}\right)p^L$,[131]

$$1-\Upsilon^0_t=\left(1-\phi^{\text{nash}}\right)+\underbrace{\left[\underbrace{\beta_{t+1}\frac{p^L_{t+1}}{p^L_t}}_{\text{exogenized}}\right]\left[\frac{D^n_{t+1}}{\beta_{t+1}}\right]}_{\text{average over sectors}}\cdot\Upsilon^1_{t+1}$$

where the Υ^i_t are steady state expressions of the respective derivatives.

The structural finding rate is given by:

$$x^*_t=1-\frac{1}{1+v^*_t/n^{s*}_t}$$

where v^*_t is the endogenous structural number of job postings. There is also the equilibrium condition where $mv=xn^s$ as well as the necessary population aggregation relationships.

Individual search is given by the household first order condition:

$$\left[1-r^b_{a,t}-\frac{1}{1+\eta^h}\right]^{\eta^n/\eta^b}=\frac{\lambda^{n*}_{a,t}\left[q^{s*}_{a,t}\right]^{\eta^n}}{x^*_t}-(1-\delta_{a+1,t+1})\quad\underbrace{s_a\beta_{a,t}\frac{Z^S_{a+1}}{Z^S_a}\frac{\rho^e_{a+1}}{\rho^e_a}}_{\text{discount factor (exogenized)}}\quad\frac{\lambda^{n*}_{a+1,t+1}\left[q^{s*}_{a+1,t+1}\right]^{\eta^n}}{x^*_{t+1}}$$

where structural objects replicate their respective model counterparts. The above structural equations provide a closed solution for structural employment.

In principle, MAKRO can be calibrated solely on the basis of data for the most recent data year. The age-distributed parameter for the disutility of search is calibrated to hit actual employment. Using this parameter then yields actual and structural employment both in recent data years and in the future. However, the Ministry of Finance has a dedicated projection of the age-distributed structural employment based on register data, which satisfies their principles for calculating structural employment. This is then reflected in the calibrated structural parameter for disutility of search. In order to make structural and actual employment converge, the actual and structural parameter of labor market participation must converge in the long run.

9.2 Structural GVA

Structural GVA, Y^*_t, is an aggregate economy measure which is calculated as:

$$Y^*_t=A^*_t\left(L^*_t\right)^\alpha K^{1-\alpha}_t+Y^{Exo}_t$$

where A^*_t is structural TFP, L^*_t is structural employment (as calculated above) and K_t is the actual (realized) capital stock all in sectors (and therefore affected by the business

[131]This is an approximation, and used here as an intuitive explanation to the solution to the structural model.

cycle), and Y_t^{Exo} is GVA in sectors not affected by the business cycle. The public sector is exogenous and by construction not affected by the business cycle and is part of Y_t^{Exo}. The extraction sector and the housing sector which mostly consist of capital, and sea transport and aggriculture where demand is mostly driven by factors exogenous to the domestic business cycle, are also included in Y_t^{Exo}. This methodology is based on the same principles as the calculation of potential/structural GVA by the Ministry of Finance.[132]

Structural TFP, A_t^*, is calculated on the basis of the following equation:

$$Y_t = A_t^* \left(u_t^L L_t\right)^{\alpha} \left(u_t^K K\right)_t^{1-\alpha} + Y_t^{Exo}$$

where both output ($Y =$GVA), labor and capital are the realized values. u_t^L and u_t^K denote capacity utilization for labor and capital, which are set at unity for this calculation.

[132] "Finansministeriets beregning af gab og strukturelle niveauer" (Finansministeriet, November 2020).

10 The Input-Output system

The Input/Output matrix organizes market clearing conditions, equating the demand and supply of goods and services. We follow the National Accounting classification where aggregate demand consists of private, C, and public, G, consumption, investment, I, exports, X, and of material inputs into production, R. This demand is met by domestic production, Y, and by imports, M.

For both households and firms, the two bottom levels of the CES demand tree can be viewed as independent zero profit intermediary sectors. This is where consumption goods, investment goods, and intermediate inputs, are sourced from the different production sectors, and from home and abroad. The specific organization of our firm and household demand trees, and the specificity of the composition of export goods, requires the input-output structure to map the decomposition of the demand bundles with the eight-sector decomposition of private production. To have an idea of the size of the system, each of the 8 private production sectors plus the public production sector can potentially demand intermediate inputs and investment inputs from each other yielding 2×81 columns.

Each quantity demand object $(q^R_{r,t}, q^C_{c,t}, q^G_{g,t}, q^I_{k,t})$ is produced (CES assembled) at an upper level with inputs from potentially all sectors, and at a lower level using both domestic production and imports. The lower level inputs from the domestic sector are called $q^{IOy}_{d,s,t}$, while inputs imported are, $q^{IOm}_{d,s,t}$. These entries into the input/output system have three subscripts. The set d identifies the demand side and consists of the sets r, c, g, k and x.[133] The supply side index identifies the production sector s. Demand side d demands output from supply sector s. Domestic and foreign supplies aggregate with a CES function into $q^{IO}_{r,s,t} = CES(q^{IOy}_{r,s,t}, q^{IOm}_{r,s,t})$. All of these have an extra IO label in the code.

Table 1 shows an Input/Output table where the demand components are column vectors and the supply ones are row vectors. An object such as $q^{IOy}_{r,s,t}$ represents a sum of r columns. As Table 1 considers 2 sectors only, the object $q^{IOy}_{r,1,t} = q^{IOy}_{r=1,1,t} + q^{IOy}_{r=2,1,t}$ is the amount of output from production sector $s = 1$ allocated to satisfy the demand for materials r from sectors 1 ($r = 1$) and 2 ($r = 2$). We only see the sum $q^{IOy}_{r,1,t}$, not the two sub objects that compose it.[134]

The consumer and firm chapters contain a partial discussion of the subject of this chapter. It is there that we first describe the decomposition of demand by sectors as proportional (Leontief) with the lower level decomposition across domestic and foreign suppliers having a non zero elasticity. In Table 1 this means the ratio $q^{IOy}_{j,1,t}/q^{IOy}_{j,2,t}$ is an exogenous constant while the ratio $q^{IOy}_{j,1,t}/q^{IOy}_{j,1,t}$ reacts endogenously to relative prices. It also means we write the aggregator for domestic and foreign sources in a single supply sector ($s = 1$) as $q^{IO}_{r,1,t} = CES(q^{IOy}_{r,1,t}, q^{IOm}_{r,1,t})$ and in the level above with a Leontief aggregator over supply sectors we have $q^R_{r,t} = LFF_s(q^{IO}_{r,s,t})$.

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[133] The investment index set i contains the set k plus the index for inventories which are treated differently from equipment and structures. Most of what we discuss applies only to the set k.

[134] There is a subtle detail here: Instead of extra labeling with an R, as in $q^{Ry}_{d,s,t}$ we map the set d into the set r to define the type of use we give to the goods from sector s, which means we need only the label $q^y_{r,s,t}$.

Table 10.1: Input-Output Matrix. 2 Sector Example

		Demand aimed at domestic and foreign suppliers							
		$q^{R}_{r,t}$		$q^{C}_{c,t}$		$q^{I}_{k,t}$		$q^{X}_{x,t}$	
		$q^{IO}_{r,s,t}$		$q^{IO}_{c,s,t}$		$q^{IO}_{k,s,t}$		$q^{IO}_{x,s,t}$	
Supply		D	F	D	F	D	F	D	F
Domestic	$Y_{1,t}$	$q^{IOy}_{r,1,t}$		$q^{IOy}_{c,1,t}$		$q^{IOy}_{i,1,t}$		$q^{IOy}_{x,1,t}$	
$Y_{s,t}$	$Y_{2,t}$	$q^{IOy}_{r,2,t}$		$q^{IOy}_{c,2,t}$		$q^{IOy}_{i,2,t}$		$q^{IOy}_{x,2,t}$	
Foreign	$M_{1,t}$		$q^{IOm}_{r,1,t}$		$q^{IOm}_{c,1,t}$		$q^{IOm}_{i,1,t}$		$q^{IOm}_{x,1,t}$
$M_{s,t}$	$M_{2,t}$		$q^{IOm}_{r,2,t}$		$q^{IOm}_{c,2,t}$		$q^{IOm}_{i,2,t}$		$q^{IOm}_{x,2,t}$

Each column represents a horizontal sum of s columns.

10.1 Market clearing prices

In MAKRO the most disaggregated production level is the sectoral level indexed s. All output from a sector s has the same price (before taxes) irrespective of who buys it.[135]

The after tax price may vary depending on the buyer as indirect taxes can vary across demand components. For example, households generally face higher indirect taxes on private consumption than firms do on material inputs. Demand prices (paid by the buyer) are then:

$$P^{IOy}_{d,s,t} = \left(1 + \tau^{IOy}_{d,s,t}\right) P^{Y}_{s,t}$$

$$P^{IOm}_{d,s,t} = \left(1 + \tau^{IOm}_{d,s,t}\right) P^{M}_{s,t}$$

where $P^{Y}_{s,t}$ and $P^{M}_{s,t}$ are the prices received by producers which are the same irrespective of the identity of the buyer.

The indirect tax rates for domestic production and imports are compositions of customs, net duties, and valued added tax rates:

$$\tau^{IOy}_{d,s,t} = \left(1 + \tau^{NDy}_{d,s,t}\right)\left(1 + \tau^{Vaty}_{d,s,t}\right) - 1$$

$$\tau^{IOm}_{d,s,t} = \left(1 + \tau^{Cus}_{d,s,t}\right)\left(1 + \tau^{NDm}_{d,s,t}\right)\left(1 + \tau^{Vatm}_{d,s,t}\right) - 1$$

where $\tau^{Cus}_{d,s,t}$ are the custom rates and $\tau^{Vatj}_{d,s,t}$ are the VAT rates, all exogenous to the model and taken from the Input-/Output data table. Net duty rates, $\tau^{NDj}_{d,s,t}$, are also taken from the Input-/Output table and consist of rates on gross duties τ minus gross subsidies $\wr$:[136]

$$\tau^{NDy}_{d,s,t} = \tau^{Dy}_{d,s,t} - \wr^{Dy}_{d,s,t}$$

$$\tau^{NDm}_{d,s,t} = \tau^{Dm}_{d,s,t} - \wr^{Dm}_{d,s,t}$$

In the model the gross duty and subsidy rates are exogenous. They are imputed in order to ensure all gross duties are positive, and also that disaggregated duty rates give the value of total subsidies when they are aggregated. All rates for duties, subsidies,

[135] In the Input-Output tables from the national accounts it is possible to derive prices for the different I-O cells. In these cells the net price from each delivering sector will vary. We disregard this information as to make the model more tractable. In the ADAM model prices are not explicitly defined for the I-O cells. Instead, they use constant I-O coefficients in determining the aggregate price, which implicitly assumes all output from sector s has the same price irrespective of who buys it.

[136] Three car registration taxes, $\tau^{Reg}_{d,t}$ for $d = cBil, g, iM$, are explicit (in addition to gross duties).

customs and VAT are allowed to vary both across the demand and supply sectors. In the most disaggregated national accounts and in ADAM they are identical for all deliveries s. The variation on this dimension in MAKRO relative to ADAM is due to the fact that production sectors here aggregate a higher number of subsectors.

10.2 Demand trees

For an agent purchasing a given good d, this good d is a composition of different goods produced in the different sectors s, and just below the contribution of goods from sector s to good d, there are sector s components produced domestically, $q^{IOy}_{d,s,t}$, and sector s components which are imported, $q^{IOm}_{d,s,t}$. Markets exist only at the very bottom of the tree. And it is here that prices are determined by market equilibrium.

10.2.1 The bottom of the demand tree. Standard CES problem.

This decomposition of imported and domestic quantities is at the bottom of the demand tree. The standard way of solving the bottom problem aggregates quantities using a CES aggregator with a fixed elasticity of substitution $\eta \equiv \eta^{IO}_{d,s}$:

$$q^{IO}_{d,s,t} = \left(\left(\mu^{IOy}_{d,s,t}\right)^{\frac{1}{\eta}}\left(q^{IOy}_{d,s,t}\right)^{\frac{\eta-1}{\eta}} + \left(\mu^{IOm}_{d,s,t}\right)^{\frac{1}{\eta}}\left(q^{IOm}_{d,s,t}\right)^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}$$

and the model solves with the two demand functions and the zero profit constraint (so that the CES aggregator function is never used)

$$q^{IOy}_{d,s,t} = \mu^{IOy}_{d,s,t} \cdot q^{IO}_{d,s,t} \cdot \left(\frac{P^{IOy}_{d,s,t}}{P^{IO}_{d,s,t}}\right)^{-\eta^{IO}_{d,s}}$$

$$q^{IOm}_{d,s,t} = \mu^{IOm}_{d,s,t} \cdot q^{IO}_{d,s,t} \cdot \left(\frac{P^{IOm}_{d,s,t}}{P^{IO}_{d,s,t}}\right)^{-\eta^{IO}_{d,s}}$$

$$P^{IO}_{d,s,t}q^{IO}_{d,s,t} = P^{IOy}_{d,s,t} \cdot \left(q^{IOy}_{d,s,t}\right) + P^{IOm}_{d,s,t} \cdot \left(q^{IOm}_{d,s,t}\right)$$

where $P^{IO}_{d,s,t}$ is the corresponding zero profit CES price aggregate of prices $\left(P^{IOy}_{d,s,t}, P^{IOm}_{d,s,t}\right)$. Typically we use one demand function and the ratio of the two which is

$$\frac{q^{IOm}_{d,s,t}}{q^{IOy}_{d,s,t}} = \frac{\mu^{IOm}_{d,s,t}}{\mu^{IOy}_{d,s,t}} \cdot \left(\frac{P^{IOm}_{d,s,t}}{P^{IOy}_{d,s,t}}\right)^{-\eta^{IO}_{d,s}}$$

10.2.2 The bottom of the demand tree. MAKRO adaptation.

We tweak the above structure by changing the price that affects demand and rewrite the entire set up as follows. We use explicitly the CES aggregator

$$q^{IO}_{d,s,t} = \left(\left(\mu^{IOy}_{d,s,t}\right)^{\frac{1}{\eta}}\left(q^{IOy}_{d,s,t}\right)^{\frac{\eta-1}{\eta}} + \left(\mu^{IOm}_{d,s,t}\right)^{\frac{1}{\eta}}\left(q^{IOm}_{d,s,t}\right)^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}$$

and then use the ratio

$$\frac{q^{IOm}_{d,s,t}}{q^{IOy}_{d,s,t}} = \frac{\mu^{IOm}_{d,s,t}}{\mu^{IOy}_{d,s,t}} \cdot \left(R^{IOym}_{d,s,t}\right)^{-\eta^{IO}_{d,s}}$$

where we have a slow moving price object

$$R^{IOym}_{d,s,t} = \left(R^{IOym}_{d,s,t-1}\right)^{\lambda_{d,s}} \left(\frac{P^{IOm}_{d,s,t}}{P^{IOy}_{d,s,t}}\right)^{1-\lambda_{d,s}}$$

and then finally the constraint

$$P^{IO}_{d,s,t} q^{IO}_{d,s,t} = P^{IOy}_{d,s,t} \cdot \left(q^{IOy}_{d,s,t}\right) + P^{IOm}_{d,s,t} \cdot \left(q^{IOm}_{d,s,t}\right)$$

which defines the price $P^{IO}_{d,s,t}$. This is no longer the standard CES price but nevertheless the price consistent with zero profits.

This structure is imposed to generate a dampening of the reaction of quantities to price movements in the short run.

10.2.3 One step up the demand tree

Here we have a Leontief allocation as in this level the composition is in fixed proportions. For $d = \{r, c, k, x\}$ we have $q^{IO}_{d,s,t} = \mu^{IO}_{d,s,t} q^{D}_{r,t}$. In this expression $\mu^{IO}_{d,s,t}$ are calibrated parameters and the quantities $q^{D}_{r,t}$ are determined by the optimal input decisions of firms, and optimal consumption decisions of households. For each demand side object we have then:[137]

$$q^{IO}_{r,s,t} = \mu^{IO}_{r,s,t} \cdot q^{R}_{r,t}$$

$$q^{IO}_{c,s,t} = \mu^{IO}_{c,s,t} \cdot q^{C}_{c,t}$$

$$q^{IO}_{k,s,t} = \mu^{IO}_{k,s,t} \cdot q^{I}_{k,t}$$

$$q^{IO}_{x,s,t} = \mu^{IO}_{x,s,t} \cdot q^{X}_{x,t}$$

For example $\mu^{IO}_{c,s,t}$ is the fraction of total consumption demand $q^{C}_{c,t}$ that falls on goods produced or imported by sector s. There is one detail in these four expressions: the expression for investment is not identical to the other ones because the index $d = k$ identifies the type of investment and not the sector demanding that investment good. The reason is that the organization of investment is identical for all demand sectors so the sector origin index is dropped.

Regarding prices, at this level in the tree we aggregate over the s sectors to obtain the good actually desired by the buying agent. Prices satisfy

$$P^{R}_{r,t} = \frac{\sum_s P^{IO}_{r,s,t} q^{IO}_{r,s,t}}{q^{R}_{r,t}} = \frac{\sum_s P^{IO}_{r,s,t} \mu^{IO}_{r,s,t} \cdot q^{R}_{r,t}}{q^{R}_{r,t}} = \sum_s P^{IO}_{r,s,t} \mu^{IO}_{r,s,t}$$

$$P^{C}_{c,t} = \frac{\sum_s P^{IO}_{c,s,t} q^{IO}_{c,s,t}}{q^{C}_{c,t}} = \frac{\sum_s P^{IO}_{c,s,t} \mu^{IO}_{c,s,t} \cdot q^{C}_{c,t}}{q^{C}_{c,t}} = \sum_s P^{IO}_{c,s,t} \mu^{IO}_{c,s,t}$$

$$P^{I}_{k,t} = \frac{\sum_s P^{IO}_{k,s,t} q^{IO}_{k,s,t}}{q^{I}_{k,t}} = \frac{\sum_s P^{IO}_{k,s,t} \mu^{IO}_{k,s,t} \cdot q^{I}_{k,t}}{q^{I}_{k,t}} = \sum_s P^{IO}_{k,s,t} \mu^{IO}_{k,s,t}$$

[137]Inventory investments which are in set i are not in the set k as they are determined by a different equation.

$$P_{x,t}^{X}=\frac{\sum_s P_{x,s,t}^{IO}q_{x,s,t}^{IO}}{q_{x,t}^{X}}=\frac{\sum_s P_{x,s,t}^{IO}\mu_{x,s,t}^{IO}\cdot q_{x,t}^{X}}{q_{x,t}^{X}}=\sum_s P_{x,s,t}^{IO}\mu_{x,s,t}^{IO}$$

This is a general rule. There are exceptions which we discuss below.[138]

10.2.4 Import to reexport

As there is no substitution between direct exports and export to re-imports - these relationships are not valid for d=x. Instead, for exports we define two aggregate quantities for imported and domestic inputs at this level:

$$q_{x,s,t}^{y}=\mu_{x,s,t}^{IOXy}\cdot q_{x,t}^{Xy}$$
$$q_{x,s,t}^{m}=\mu_{x,s,t}^{IOXm}\cdot q_{x,t}^{Xm}$$

10.3 Aggregates

The production of each sector is the sum of deliveries to all demand components for both domestic production and imports:

$$Y_{s,t}=\sum_d q_{d,s,t}^{IOy}$$
$$M_{s,t}=\sum_d q_{d,s,t}^{IOm}$$

These objects have well defined prices since the production of each sector has an equilibrium price. However, as we move one step up in aggregation summing over s, prices and quantities require a definition because we are summing over different objects.

The demand side aggregate quantities (R_t, G_t, I_t, X_t) and the supply components (Y_t, M_t), with respective prices $(P_t^R, P_t^G, P_t^I, P_t^X)$ and (P_t^Y, P_t^M) , have no theoretical price index or quantity aggregator as they are not supported by a model driven CES technology or preference aggregator.[139] Therefore we use Paasche price indices and Laspeyres indices for the corresponding quantities. Using a generic name $Z=R,G,I,X,Y,M$, we first have the definition

$$P_t^Z Z_t=\sum_d P_{d,t}^Z Z_{d,t}$$

and then we add the index relationship

$$Z_t P_{t-1}^Z=\sum_d P_{d,t-1}^Z Z_{d,t}$$

and together they imply the price and quantity dynamic indices

$$Z_t=Z_{t-1}\frac{\sum_d P_{d,t-1}^Z Z_{d,t}}{\sum_d P_{d,t-1}^Z Z_{d,t-1}}\quad\text{and}\quad P_t^Z=P_{t-1}^Z\frac{\sum_d P_{d,t}^Z Z_{d,t}}{\sum_d P_{d,t-1}^Z Z_{d,t}}$$

[138] This does not apply to $P_{c,t}^C$ due to the way we handle tourism. For exports $\mu_{x,s,t}^{IO}$ is not defined.

[139] We do the same for GDP and for aggregate gross value added. We note that housing and non-housing consumption do not face this problem as they have a model-defined aggregate prices. Investment quantities aggregate linearly so that we do not need a price index to calculate the price of total investment of a given type (buildings or equipment). Using the index approach does not affect the outcome in a significant way.

These equations simplify in the case where quantities are homogeneous as price indices become unnecessary. In such a case we replace the index equation with the quantity sum and work with

$$P_t^Z Z_t = \sum_d P_{d,t}^Z Z_{d,t} \text{ and } Z_t = \sum_d Z_{d,t}$$

In the data period the supply prices $P_{s,t}^Y$ and $P_{s,t}^M$ match their corresponding Paasche chain indices from national accounts. These indices equal 1 in the base year just as those for demand component prices. The corresponding quantities are therefore indexed at gross prices whereas the prices from the lower nest are indexed at net prices. This is only a level shift which is captured in the calibrated share parameter.[140] The development in both quantities is net of customs, duties and VAT.

10.4 Investment

The demand for Investment goods is detailed in the firms chapter. Firms decide on the optimal level of capital stock one period in advance due to time to build. This results in the decision of optimal current investment q_t^I so that $K_t = (1-\delta_k)K_{t-1} + q_t^I$. We assume the contributions from supplying sectors s to a unit of a given type k of capital investment $q_{k,t}^I$, are identical in all demand sectors d. If a demand sector (agriculture) wants to accumulate its stock of equipment ($k = iM$) it uses contributions from output from all sectors s to make one unit of investment in equipment. The same decomposition happens if the sector investing is manufacture, or any other sector. Not only that, the contributions from domestic and foreign sources in the lowest level of the demand tree are also identical for all sectors. This implies the price of a unit of given type of capital k is the same across sectors, $P_{k,t}^I$. It also implies quantities are constructed in the same way in all sectors and can be added across sectors to obtain aggregate demand for an investment good.

The construction is then that we have prices with the following indices, $P_{k,s,t}^{IOy}$ and $P_{k,s,t}^{IOm}$, instead of having prices with the additional demand side index $P_{k,d,s,t}^{IOy}$ and $P_{k,d,s,t}^{IOm}$. However, in the national accounts investment prices for the same capital goods differ across sectors. Therefore sectoral investment quantities in MAKRO would not match national accounts data. We fix this by adding to the model a correction factor $\lambda_{k,d,t}^{PI}$ on sectoral prices to obtain $P_{k,d,t}^I = \lambda_{k,d,t}^{PI} P_{k,t}^I$.[141] This price is then the relevant price for the optimal dynamic investment decision in each sector and is the price that enters the capital goods Euler equations which define the user cost. This factor $\lambda_{k,d,t}^{PI}$ enters after the two bottom CES constructions are decided. It affects aggregate prices linearly (we explain this below). Next we show how this is equivalent to incorporating the factor λ at the very bottom when the choice between domestic sources or imports is made.

10.4.1 Bottom

We introduce the factor λ next to the bottom prices. We also abuse notation throughout as we should have an extra index in a number of variables for the demand sector d but that extra index is omitted. As above, we have for $d = \{k\} = \{iM, iB\}$ and with $\eta_k = \eta_{k,s}^{IO}$:

[140] On the demand side prices include taxes. On the supply side they do not. IO prices are demand prices. At the bottom level, demand prices are given by supply prices plus taxes and are not standardized at 1 in the base year. At the next level, prices are normalized at 1. This process is captured in $\lambda_{d,s,t}^{IO}$ as described below in section 7.

[141] In the code $pI\_s_{[k,s,t]} = fpI\_s_{[k,s,t]} * pI_{[k,t]}$ where $s = d$.

$$q^{IO}_{k,s,t} = \left(\left(\mu^{IOy}_{k,s,t}\right)^{\frac{1}{\eta_k}}\left(q^{IOy}_{k,s,t}\right)^{\frac{\eta_k-1}{\eta_k}} + \left(\mu^{IOm}_{k,s,t}\right)^{\frac{1}{\eta_k}}\left(q^{IOm}_{k,s,t}\right)^{\frac{\eta_k-1}{\eta_k}}\right)^{\frac{\eta_k}{\eta_k-1}}$$

with the respective CES price (remember that the buyer pays taxes so we have the bottom IO buyer prices),

$$P^{CESD}_{k,s,t} = \left(\mu^{IOy}_{k,s,t}\left(\lambda_d P^{IOy}_{k,s,t}\right)^{1-\eta_k} + \mu^{IOm}_{k,s,t}\left(\lambda_d P^{IOm}_{k,s,t}\right)^{1-\eta_k}\right)^{\frac{1}{1-\eta_k}}$$

$$= \lambda_d\left(\mu^{IOy}_{k,s,t}\left(P^{IOy}_{k,s,t}\right)^{1-\eta_k} + \mu^{IOm}_{k,s,t}\left(P^{IOm}_{k,s,t}\right)^{1-\eta_k}\right)^{\frac{1}{1-\eta_k}} \equiv \lambda_d P^{CES}_{k,s,t}$$

which is the result of the zero profit condition

$$\lambda_d P^{CES}_{k,s,t} q^{IO}_{k,s,t} = \lambda_d P^{IOy}_{k,s,t} \cdot q^{IOy}_{k,s,t} + \lambda_d P^{IOm}_{k,s,t} \cdot q^{IOm}_{k,s,t}$$

which can be written without an explicit λ_d

$$P^{CES}_{k,s,t} q^{IO}_{k,s,t} = P^{IOy}_{k,s,t} \cdot q^{IOy}_{k,s,t} + P^{IOm}_{k,s,t} \cdot q^{IOm}_{k,s,t}$$

Demand side optimization generates demand functions

$$q^{IOy}_{k,s,t} = \mu^{IOy}_{k,s,t} \cdot q^{IO}_{k,s,t} \cdot \left(\frac{\lambda_d P^{IOy}_{k,s,t}}{P^{CESD}_{k,s,t}}\right)^{-\eta_k} = \mu^{IOy}_{k,s,t} \cdot q^{IO}_{k,s,t} \cdot \left(\frac{P^{IOy}_{k,s,t}}{P^{CES}_{k,s,t}}\right)^{-\eta_k}$$

and the same for $q^{IOm}_{k,s,t}$. The second equality reflects that fact that the price ratio in the demand functions does not depend on λ_d.

At this point, λ_d has disappeared. But this cannot be right since for example higher prices must imply lower quantities. That is exactly correct. The entire point is that the lower level demand quantities $q^{IOy}_{k,s,t}$ and $q^{IOm}_{k,s,t}$ are a derived demand from the quantity above, $q^{IO}_{k,s,t}$. It is that upper quantity that will reflect the effect of λ_d.

10.4.2 Next level

Above the import versus domestic production level, sectoral inputs aggregate linearly

$$q^{IO}_{k,s,t} = \mu^{IO}_{k,s,t} q^{I}_{k,t}$$

and again we emphasize that there is no demand side index d on the factor $\mu^{IO}_{k,s,t}$.[142] The index k here is only the index of which type of capital (equipment or structures) the equations are describing.[143] The price at this level of the tree, $P^{I}_{k,t}$, is given by

$$P^{I}_{k,t} q^{I}_{k,t} = \sum_s q_{k,s,t} P^{CESD}_{k,s,t} = q^{I}_{k,t} \sum_s \mu^{IO}_{k,s,t} \lambda_d P^{CES}_{k,s,t} = \lambda_d q^{I}_{k,t} \sum_s \mu^{IO}_{k,s,t} P^{CES}_{k,s,t}$$

which becomes

$$P^{I}_{k,t} = \lambda_d \sum_s \mu^{IO}_{k,s,t} P^{CES}_{k,s,t}$$

[142] The μ factors sum approximately to 1.

[143] Of course the quantities themselves depend on how much each demand sector is investing, but we do not see the demand sector index here as we will make use only of the aggregate quantity of investment.

so that the factor λ_d jumps over the aggregation across sectors. Of course, this implies the left hand side variable now requires and extra index:

$$P^I_{k,d,t} = \lambda_d \sum_s \mu^{IO}_{k,s,t} P^{CES}_{k,s,t} \equiv \lambda_d \bar{P}^{CES}_{k,t}$$

but it crucially also implies we can ignore completely the factor λ_d when we solve the two lower levels of the CES demand tree.

10.4.3 Aggregate investment

Even though prices differ by buying sector, quantities are, at the bottom of the tree, constructed identically for all buying sectors. This allows the construction of an aggregate investment price for equipment (machinery) or for structures (buildings) by averaging over the buying sectors. For investments of type k we have then:

$$P^I_{k,t} = \frac{\sum_d P^I_{k,d,t} q^I_{k,d,t}}{q^I_{k,t}} = \frac{\sum_d \lambda_d q^I_{k,d,t}}{\sum_d q^I_{k,d,t}} \bar{P}^{CES}_{k,t}$$

where in the code the demand side index is, because of the identity mapping, shown as $d = s$.

When we aggregate different types of investments, we are then required to use a price and quantity index method.

10.4.4 Inventory investment

We assume that all inventory investment in a sector comes from its own production. In the code

$$q^{IO}_{'iL',s,t} = q^{I\_s}_{'iL',s,t}$$

where the nominal amount of inventory investment $p^{I\_s}_{'iL',s,t} q^{I\_s}_{'iL',s,t}$ is an exogenous fraction of nominal output from that own sector s.

10.5 Data and calibration

Our sectoral aggregation and the resulting input-output matrix matches the corresponding nominal aggregation from the Danish National Accounts. The data for the current version of the model is, however, based on the data bank from the ADAM-model. ADAM has 12 sectors, 8 private consumption groups, 1 government consumption group, 5 investments groups and 8 export groups. There is a direct mapping from ADAM's to MAKRO's consumption, investment and export groups. This mapping is as follows:

The production sector decomposition is almost a one to one mapping from ADAM to MAKRO. Agriculture (lan,a), construction (byg,b), extraction (udv,e) housing (bol,h), sea transport (soe,qs) are identical. Energy is decomposed in two (Energy manufacturing ne and Energy refinery ng) in ADAM but joined in MAKRO (ene,ne+ng). Manufacturing is also decomposed (food nf and other nz) in ADAM and joined in MAKRO (fre,nf+nz). The private service sector in MAKRO is defined as all services including public and financial services, and excluding all public services (offentlig forvaltning og service, o1 in ADAM). This yields the mapping for services (tje,qf+qz+o-o1), and for public services (off,o1).

The MAKRO classification defines the public sector in a manner relevant to the ministries. One disadvantage is that there is some public production in each sector and taking it all from services is only an approximation. Another is that there is no information on the input-structure from and to this definition of the public sector. This is solved by assuming that material inputs to public production (off) are proportional to that of sector "o" in ADAM and by assuming that all deliveries from the public sector go to public sales, public direct investments, and public consumption. All public sales are assumed to go to private consumption of services and all public direct investments are assumed to go to intellectual rights placed under machinery investments - ie. there are no public exports and no material inputs from the public to the private sectors. These assumptions are discussed in the public production sector.

In ADAM it is assumed that, in every purchasing sector, investment in a given type of capital good contains the same input contributions from supplying sectors. National accounts data contains detailed information about the deliveries to investment types in the different sectors. MAKRO has the same assumption, mostly so as to reduce the dimensionality of the Input/Output system. This does not change the number of markets that have to clear as that is determined by the overall number of production sectors. But it reduces the number of CES tree prices and quantities that have to be computed. All sectors then have the same price index for investments.

Imports in ADAM are divided into product groups, whereas here they are a result of the consumption and production decompositions. We include energy imports (from SITC Group 3) under imports from the foreign energy industry, other imports of goods under the foreign manufacturing industry, and service imports under the foreign service industry. All imports come from these 3 industries. This means that all substitution is in relation to domestic production of goods, services and energy. Energy is exogenous so it has no endogenous substitution. Many $\mu^{IOm}_{d,s,t}$ parameters are therefore zero.[144]

In the industry-disaggregated data from the National Accounts IO tables, imports from construction, extraction, housing and public services are extremely small. However, there are imports from foreign agriculture and shipping. This should, in principle, substitute for these domestic industries. However, it is not obvious how, as long as we rely on ADAM data. Therefore, we follow ADAM and let them substitute manufacturing and private services instead.

Before taxes, the bottom prices in MAKRO are market clearing prices which are identical for all buyers. This is not the case in the national accounts for our level of aggregation and so the corresponding quantities are not the same in MAKRO and the national accounts. The aggregate quantities of sectoral imports and domestic production are scaled so the quantities of aggregate deliveries from all sectors and import components to specific demand components are the same in MAKRO and the national accounts.[145] Except on the assumed micro level all aggregates are calculated as Laspeyres quantity and Paasche price indices. All share parameters in the IO equations are statically calibrated so they are in accordance to MAKRO IO data.

[144] Imports in ADAM are more disaggregated than in MAKRO. They are divided into food, coal, crude oil, other raw materials, other energy, cars, ships and aircraft, as well as other manufacturing. Imports of food inputs are substitutes for domestic food industry output (in MAKRO part of manufacturing). Imports of other raw materials are substitutes for manufacturing in ADAM (as in MAKRO), and manufacturing imports substitutes itself. In MAKRO other import groups do not substitute for domestic production. Ships and aircraft have no significant size and cars are included primarily as input for car consumption, where the import share is so large that substitution is insignificant. However, in ADAM it matters as they do not have substitution at the disaggregated IO cell level but at the overall import group level.

[145] The imputation of data using this assumption is made in the iodata_ADAM.gms file.

10.6 Balancing share parameters

The exogenous share parameters, $\mu^{IO}_{d,s,t}$, $\mu^{IOy}_{d,s,t}$ and $\mu^{IOm}_{d,s,t}$ are constructed using the auxiliary exogenous variables $\mu^{IO_0}_{d,s,t}$, $\mu^{IOy_0}_{d,s,t}$, $\mu^{IOm_0}_{d,s,t}$, $\lambda_{d,t}$, and $\lambda^{IO}_{d,s,t}$ as follows:[146]

$$\mu^{IO}_{d,s,t} = \lambda_{d,t} \frac{\mu^{IO_0}_{d,s,t}}{\sum_s \mu^{IO_0}_{d,s,t}}$$

$$\mu^{IOy}_{d,s,t} = \lambda^{IO}_{d,s,t} \frac{\mu^{IOy_0}_{d,s,t}}{\mu^{IOy_0}_{d,s,t} + \mu^{IOm_0}_{d,s,t}}$$

$$\mu^{IOm}_{d,s,t} = \lambda^{IO}_{d,s,t} \frac{\mu^{IOm_0}_{d,s,t}}{\mu^{IOy_0}_{d,s,t} + \mu^{IOm_0}_{d,s,t}}$$

for $D = R, C, G, I, X$, $d = r, c, g, i, x$.

In the calibration $\mu^{IO}_{d,s,t}$, $\mu^{IOy}_{d,s,t}$ and $\mu^{IOm}_{d,s,t}$ are determined as usual. It is imposed that $\sum_s \mu^{IO_0}_{d,s,t} = 1$ and $\mu^{IOy_0}_{d,s,t} + \mu^{IOm_0}_{d,s,t} = 1$. Then we have

$$\mu^{IO}_{d,s,t} = \lambda_{d,t}\mu^{IO_0}_{d,s,t}$$

$$\mu^{IOy}_{d,s,t} = \lambda^{IO}_{d,s,t}\mu^{IOy_0}_{d,s,t}$$

$$\mu^{IOm}_{d,s,t} = \lambda^{IO}_{d,s,t}\mu^{IOm_0}_{d,s,t}$$

which implies $\lambda_{d,t} = \sum_s \mu^{IO}_{d,s,t}$ and $\lambda^{IO}_{d,s,t} = \mu^{IOy}_{d,s,t} + \mu^{IOm}_{d,s,t}$.

10.6.1 Exceptions: Public direct investments and public sales

The exogenous share parameters, $\mu^{IO}_{d,s,t}$, $\mu^{IOy}_{d,s,t}$ and $\mu^{IOm}_{d,s,t}$ are constructed using the auxiliary exogenous variables $\mu^{IO_0}_{d,s,t}$, $\mu^{IOy_0}_{d,s,t}$, $\mu^{IOm_0}_{d,s,t}$, $\lambda_{d,t}$, and $\lambda^{IO}_{d,s,t}$. There are two exceptions to this structure, and they are the share parameter for deliveries from the public sector to private consumption, $\mu^{IOy_0}_{c,s,t}$, and for deliveries from the public sector to investments, $\mu^{IOy_0}_{i,s,t}$ where $s = gov$. These are endogenously given so that, for $s = gov$, $q^C_{d,s,t}$ with $d = serv$ and $q^I_{d,s,t}$ with $d = iM$ are given in accordance to:[147]

$$p^I_{i,s,t} I_{i,s,t} = \mu^{Ig}_{i,t} V^{DIR}_t$$

$$p^C_{c,s,t} q^C_{c,s,t} = \mu^{Cg}_{c,t} V^{gsales}_t$$

This formulation ensures that the value of the sum of deliveries from the public sector to investments and private production are given by the two variables V^{DIR}_t and V^{gsales}_t. These two variables do not follow the general demand for investment and private consumption inputs. This implies that inputs from the public sector and hence public production will not be endogenously affected by private demand components.

[146] With this construction we can shock an individual deeper parameter indexed zero and the mechanics of the construction of the resulting parameters will share the initial shock through all of them.

[147] V^{DIR}_t = vOffDirInv[t]

11 References

- Anderson, James E., 1979. “A Theoretical Foundation for the Gravity Equation,” American Economic Review, 69(1), 106–116.
- Anderson, James, E., and Eric van Wincoop. 2003. "Gravity with Gravitas: A Solution to the Border Puzzle ." American Economic Review, 93 (1): 170-192.
- Ampudia, M., Cooper, R., Le Blanc, J., and Zhu, G. (2019). MPC Heterogeneity in Europe: Sources and Policy Implications.
- Armington, P. S., 1969: “A Theory of Demand for Products Distinguished by Place of Production,” International Monetary Fund Staff Papers, 16, 159–178.
- Attanasio, O., Peter Levell, Hamish Low, Virginia Sanchez-Marcos (2018). Aggregating Elasticities: Intensive and Extensive Margins of Women’s Labour Supply. June 4, 2018.
- Bernanke, B. and M. Gertler (1989): Agency Costs, Net Worth, and Business Fluctuations," American Economic Review 79, 14-31.
- Blanchard, O., and Fischer, S. (1989), Lectures on Macroeconomics. MIT Press.
- Blanchard, O. (2018). On the future of macroeconomic models. Oxford Review of Economic Policy, Volume 34, Issue 1-2, Spring-Summer 2018, Pages 43–54.
- Blanchard, O. (2017), “Should we get rid of the natural rate hypothesis?”, NBER Working Paper 24057, November 2017.
- Blanchard, O., and Galí, J., 2008. "Labor Markets and Monetary Policy: A New-Keynesian Model with Unemployment," NBER Working Papers 13897, National Bureau of Economic Research, Inc.
- Boldrin, Michelle, Lawrence J. Christiano, and Jonas Fisher “Habit Persistence, Asset Returns, and the Business Cycle,” American Economic Review, 91: 149—166, 2001. 23
- Boserup, S., Kpoczuk, W., Kreiner, C. (2016), "The Role of Bequests in Shaping Wealth Inequality: Evidence from Danish Wealth Records," American Economic Review, American Economic Association, vol. 106(5), pages 656-661, May.
- Browning, M., & Ejrnæs, M. (2009). Consumption and Children. Review of Economics and Statistics, 91(1), 93-111.
- Campbell, John Y., and Mankiw, N. Gregory, (1989). Consumption, Income and Interest Rates: Reinterpreting the Time Series Evidence. NBER Macroeconomics Annual 1989, Volume 4, (p. 185 - 246).
- Carroll, Christopher. (2001). Death to the Log-Linearized Consumption Euler Equation! (And Very Poor Health to the Second-Order Approximation). Advances in Macroeconomics. 1. 1003-1003. 10.2202/1534-6013.1003.
- Christiano, L., Eichenbaum, M., and Trabant, M., (2016), Unemployment and Business Cycles, Econometrica, July 2016, 84(4), pp. 1523-1569.
- Cooper, R., and Adda, J. (2000), “Balladurette and Juppette: A Discrete Analysis of Scrapping Subsidies”, Journal of Political Economy Volume 108, Number 4 | August 2000.

- Coibion, O., and Gorodnichenko, Y. (2013) “Is the Phillips Curve alive and well after all?”, NBER Working Paper 19598, October 2013.
- Drechsel, T. (2021). Earnings-Based Borrowing Constraints and Macroeconomic Fluctuations. University of Maryland. (2019 LSE Job Market paper)
- Eaton, J., & Kortum, S. (1999). International Technology Diffusion: Theory and Measurement. International Economic Review, 40(3), 537-570.
- Eaton, J., & Kortum, S. (2002). Technology, Geography, and Trade. Econometrica, 70(5), 1741-1779.
- Galí, J., and Gertler, M. (1999), Inflation Dynamics: A Structural Econometric Analysis. Journal of Monetary Economics Volume 44, Issue 2, October 1999, Pages 195-222.
- Galí, J., Smets, F., and Wouters, R., (2012), "Unemployment in an Estimated New Keynesian Model," NBER Macroeconomics Annual 26 (2012): 329-360.
- Gomes, J. F. (2001). Financing Investment. American Economic Review 91, 1263-1285.
- Gordon, R. (2013) “The Phillips Curve is alive and well”, NBER Working Paper 19390, August 2013.
- Gourinchas, P., and Parker, J. (2002). Consumption Over the Life Cycle. Econometrica. 70. 47-89.
- Kaplan, G., G., Mitman, K., and Violante, G. (2017), The Housing Boom and Bust: Model meets Evidence. NBER Working Paper 23694, August 2017.
- Kaplan, G., Violante, G. L., and Weidner, J., (2014). The Wealthy hand-to-mouth. NBER Working Paper 20073, April 2014.
- King, Plosser and Rebelo (1988), “Production, Growth and Business Cycles”, Journal of Monetary Economics 21, 195-232.
- Kocherlakota, N. (2010). Modern Macroeconomic Models as Tools for Economic Policy. Federal Reserve Bank of Minneapolis. Annual Report Essay May 4, 2010. Narayana Kocherlakota, President, Minneapolis Federal Reserve Bank.
- Kravik, Erling Motzfeldt og Yasin Mimir (2019). “Navigating with NEMO”.
- Krugman, Paul R., 1979. "Increasing returns, monopolistic competition, and international trade," Journal of International Economics, Elsevier, vol. 9(4), pages 469-479, November.
- Krugman, P., 1980. “Scale Economies, Product Differentiation, and the Pattern of Trade,” American Economic Review, 70(5), 950–959.
- Kiyotaki, N., and Moore, J., (1997). Credit Cycles. Journal of Political Economy Vol. 105, No. 2, pp. 211-248.
- Li, W., Liu, H., Yang, F., and Yao, R. (2016), “Housing over Time and over the Life Cycle: A Structural Estimation”, International Economic Review, Volume 57, Issue 4, Pages 1237-1260.
- Lian, C. and Yueran Ma (2020), Anatomy of Corporate Borrowing Constraints. The Quarterly Journal of Economics, vol. 136, issue 1, 229-291

- Ljungqvist, L., and Sargent, T. (2017). "The Fundamental Surplus," American Economic Review, American Economic Association, vol. 107(9), pages 2630-2665, September.
- Mankiw, G. (1982), “Hall’s Consumption Hypothesis and Durable Goods”, Journal of Monetary Economics 10, 417-425.
- Melitz, M., 2003, The impact of trade on intra-industry reallocations and aggregate industry productivity. Econometrica 71, Number 6, pp. 1695-1725.
- Mortensen, D., and Pissarides, C. (1994). Job Creation and Job Destruction in the Theory of Unemployment. The Review of Economic Studies, 61(3), 397-415.
- Mortensen, Anne Ulstrup (2015) “The Keynesian Multiplier in a CGE Model”, Master Thesis, Institute of Economics, University of Copenhagen.
- Mortensen, M., Sisay, D., and Kristensen, T., 2014. “Export market index: forecast for latest historical period”, Working Paper, Danmarks Statistik Modelgruppen.
- Pedersen, J. (2016). An Estimated DSGE Model for Denmark with Housing, Banking, and Financial Frictions, Danmarks Nationalbank Working Paper N. 108, 2016.
- Petrongolo, B., and Pissarides, C., (2001), Looking into the Black Box: A Survey of the Matching Function, Journal of Economic Literature Vol. 39, No. 2, June 2001, (pp. 390-431)
- Rebelo, S. (2005), “Real Business Cycle Models: Past, Present, and Future”, The Scandinavian Journal of Economics 107(2), 217-238.
- Rogerson, R. (1988). Indivisible Labor, Lotteries and Equilibrium. Journal of Monetary Economics 21, pp 3-16.
- Rotemberg, J., (1982). “Monopolistic Price Adjustment and Aggregate Output”. Review of Economic Studies 49(4), 517–531.
- Sisay, D., 2013. “Export Market and Market Price Indices for ADAM”, Working Paper, Danmarks Statistik Modelgruppen.
- Straathof, Bas., 2008. "Gravity with gravitas: comment," CPB Discussion Paper 111.rdf, CPB Netherlands Bureau for Economic Policy Analysis.
- Temere, Dawit S., and Tony M. Kristensen. 2016. “Supply factors in trade determination.” Danmarks Statistik Modelgruppen, Working Paper DSI080816, August 2016.
- Temere, Dawit S., and Tony M. Kristensen. 2016. “Incorporating supply effects in ADAM’s export equations.” Danmarks Statistik Modelgruppen, Working Paper DSI111116, November 2016.
- Yang, F. (2009), “Consumption over the life cycle: How different is housing?”, Review of Economic Dynamics, Volume 12, Issue 3, July 2009, Pages 423-443

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