DREAM
Danish Research Institute for
Economic Analysis and Modelling

MAKRO

Overcoming Bias in Elasticity of Substitution Estimation: A Novel Approach with Empirical Application

Christian B. Kastrup and Christian Vikkelsø

Working paper 2023:2
8 December 2023

www.dreamgruppen.dk

Overcoming Bias in Elasticity of Substitution Estimation: A Novel Approach with Empirical Application

Christian B. Kastrup and Christian Vikkelsø

Working paper
08-12-2023

www.dreamgruppen.dk

Abstract

This paper addresses the crucial issue of estimating the elasticity of substitution between production factors in macroeconomic models, with a particular focus on overcoming measurement errors in the user cost of capital. We introduce a system estimator approach that combines the relative first-order conditions and the production function to mitigate bias caused by measurement errors. Through simulations, we demonstrate that this approach yields unbiased estimates, even in the presence of misspecified technical change. Applying our method to data from MAKRO, we find a range of elasticities across sectors and production nests, ranging between 0.5 and 1.7.

1 Introduction

The elasticity of substitution between production factors is a central parameter in most macroeconomic models, and its estimation has recently regained attention due to the global decline in the labor share (Karabarbounis and Neiman, 2014; Oberfield and Raval, 2021). The estimation of the elasticity is associated with two main issues: First, the estimation requires factor price movements unrelated to technical change (the Diamond et al., 1978, non-identification result). Second, the factor price on capital (the "user-cost") is, in general, unobserved, and its measurement has either been based on a model-implied expression of the user cost or derived residually by making assumptions about the markup (Hulten, 2010). This implies that the user cost is likely measured with error which may generate an "attentuation bias" (León-Ledesma et al., 2010).

In MAKRO, we have addressed these issues in a series of papers. Kastrup et al. (2022) deviate from the existing literature by applying a flexible modeling of technical change in the estimation of the inverse first-order conditions (FOC) of the firm's maximization problem. The paper presents simulation evidence that the approach outperforms traditional estimators relying on less flexible trend assumptions. Kronborg et al. (2021) and Kronborg and Poulsen (2021) apply the framework to estimate the production factor elasticities in MAKRO. The first paper applies a user cost expression based on MAKRO, and the second is a user cost derived residually by making assumptions about the markups.

In both papers, we find that many estimates are implausibly low at close to zero and, in some cases, even negative. A potential reason for this may be due to measurement error in the user cost. León-Ledesma et al. (2010) show that measurement errors in the explanatory variable generate a downward attenuation bias in the estimated elasticity. Thus, this is, in particular, an issue for estimation frameworks based on the inverse FOC, where the variable with measurement error (the user cost) is used as the explanatory variable and demand as the dependent variable. The authors show that a system estimator consisting of both FOC and the production function is not subject to this downward bias.

In this paper, we apply a system estimator approach to deal with measurement errors in the user cost. We first apply a simulation study to show that the system estimators from León-Ledesma et al. (2010) are biased toward unity. As shown by Kastrup et al. (2022), this bias happens because the observed output is correlated with the measurement error in the user cost. Our novel and main contribution is to show that a system estimator relying on the *relative* FOC removes this bias. This is because output only influences the FOC of the production factors individually and not the *relative* FOC. Thus, a system estimator consisting of the relative FOC and the production function leads to unbiased median estimates, even when technical change is misspecified.

We next apply our new system estimator to estimate the elasticity of substitution in MAKRO. The data are obtained directly from MAKRO's databank supplied by Statistics Denmark and cover the time period 1983-2019 as well as all sectors in MAKRO except housing and public. For each sector, we estimate a nested CES production function (one nest at a time) with the same nest structure as in MAKRO. Firms apply equipment capital (K), energy (E), labor (L), structures capital (B), and intermediate inputs (R). Across sectors and nests, the elasticities range from 0.5 to 1.7, with the majority being between 0.7 and 0.95. Thus, our estimates are not subject to the same issue of a downward bias toward zero.

The paper is structured as follows: Section 2 presents the system estimation approach. In Section 3, we present the simulation study and the results. In section 4, we present the data from MAKRO and apply the system estimator to estimate the elasticities in MAKRO. We conclude in Section 5.

2 The system estimator

The firms in a given sector produce using a (((KE)L)B)R nest structure. Denoting $X_{1,t}, X_{2,t}$ as the two production factors in a given nest, the CES function is:

$$Y_t = \left[\pi \left(\alpha_{1,t} X_{1,t}\right)^{(\sigma-1)/\sigma} + (1-\pi)\left(\alpha_{2,t} X_{2,t}\right)^{(\sigma-1)/\sigma}\right]^{\frac{\sigma}{\sigma-1}} \tag{1}$$

Y_t is the produced output in the nest, $\alpha_{1,t}, \alpha_{2,t}$ the factor-augmenting productivities, and σ is the elasticity of substitution. We assume that $\alpha_{1,t} = \alpha_{1,0} e^{\gamma_1 t}$ and $\alpha_{2,t} = \alpha_{2,0} e^{\gamma_2 t}$, where $\alpha_{1,0}, \alpha_{2,0}$ denote the initial values, both set equal to unity. γ_1, γ_2 are the constant growth rate parameters of, respectively, factor one and factor two. We also apply an alternative specification of the growth rates where they are specified as Box-Cox trends (see e.g., Klump et al., 2007)[1].

As recently highlighted by León-Ledesma et al. (2010) and Klump et al. (2007), the estimation of this non-normalized production function leads to biased estimates of the substitution elasticity. Therefore, we normalize the production function as follows:

$$Y_t = \xi \bar{Y} \left[\bar{\pi} \left(e^{\gamma_1 (t-\bar{t})} \frac{X_{1,t}}{\bar{X}_1}\right)^{(\sigma-1)/\sigma} + (1-\bar{\pi}) \left(e^{\gamma_2 (t-\bar{t})} \frac{X_{2,t}}{\bar{X}_2}\right)^{(\sigma-1)/\sigma}\right]^{\frac{\sigma}{\sigma-1}} \tag{2}$$

Where a bar denotes an average over time, geometric for all except $\bar{\pi}$ and $\bar{t}$, which are, respectively, the aritmetic average of factor $X_{1,t}$ share in Y_t over time, and the average time period, i.e. $\bar{t} = T/2$. $\xi = Y_0/\bar{Y}$ is a parameter that relates the average output to the initial output. Maximizing output w.r.t. the two production factors imply the following FOC's:

$$P_{1,t} = \bar{\pi} \frac{\bar{Y}}{\bar{X}_1} \left(\frac{Y_t/\bar{Y}}{X_{1,t}/\bar{X}_1}\right)^{1/\sigma} e^{\frac{\sigma-1}{\sigma}\gamma_1 (t-\bar{t})} \xi^{\frac{\sigma-1}{\sigma}} \tag{3}$$

[1] In the normalized production function below, the Box-Cox transformation is specified as $\alpha_{1,t} = \alpha_{1,0} e^{\bar{t}\frac{\gamma_1}{\lambda_1}\left(\left(\frac{t}{\bar{t}}\right)^{\lambda}-1\right)}$. This functional form implies that when $\lambda < 1$, the growth rate is decreasing, $\lambda = 1$ is a linear trend, $\lambda > 1$ an increasing growth rate.

$$P_{2,t} = (1-\bar{\pi})\frac{\bar{Y}}{\bar{X}_2}\left(\frac{Y_t/\bar{Y}}{X_{2,t}/\bar{X}_2}\right)^{1/\sigma} e^{\frac{\sigma-1}{\sigma}\gamma_2(t-\bar{t})}\xi^{\frac{\sigma-1}{\sigma}} \tag{4}$$

Combining equation (3) and (4) yields the relative FOC:

$$\frac{P_{1,t}}{P_{2,t}} = \frac{\bar{\pi}}{1-\bar{\pi}}\frac{\bar{X}_2}{\bar{X}_1}\left(\frac{X_{2,t}/\bar{X}_2}{X_{1,t}/\bar{X}_1}\right)^{1/\sigma} e^{\frac{\sigma-1}{\sigma}(\gamma_1-\gamma_2)(t-\bar{t})} \tag{5}$$

The standard features of CES systems apply: The relative marginal product depends on the relative input of the two production factors and their relative productivity. We estimate two different versions of the system estimator. The first is the standard system estimator from, e.g., León-Ledesma et al. (2010, 2015)[2], which consist of estimating the simultaneous system of equation (2)-(4), all in logs. The second system estimator consists of estimating the relative FOC (5) simultanously with the production function (2), both in logs. This system estimator is the main novelty of this paper. The primary gain from using the relative FOC is because it is unaffected by output (see equaton 5). This eliminates the endogeneity bias that may be present if output is correlated with the error term. For both estimations, we use a Feasible Generalized Non-linear Least Squared estimator (FGNLS).[3]

3 Simulation study

In this section, we display the results from the simulation study. We compare the performance of the two different versions of the system estimators where the first estimates the two FOC and the production function simultaneously and the second the relative FOC and the production function.

3.1 Simulation setup

The simulation study is identical to Kastrup et al. (2022). We outline the simulation setup here and refer to that paper for a more detailed description of the simulation methodology. The simulation study considers the substitution between capital and labor and is calibrated to match the US data for the time period 1970-2019.[4] Throught a superscript $\star$ implies an equilibrium/theoretically consistent outcome. Note that this may deviate from the observed values in the data. As pointed out in Kastrup et al. (2022), this has important implications if the observed variables (e.g., the user cost) have measurement errors.

Firms maximization problem. Firms apply capital, K, and labor, L, to produce value-added, Y^*:

$$Y_t^* = \left[\pi\left(\alpha_{K,t}K_t\right)^{(\sigma-1)/\sigma} + (1-\pi)\left(\alpha_{L,t}L_t\right)^{(\sigma-1)/\sigma}\right]^{\frac{\sigma}{\sigma-1}} \tag{6}$$

π is the average capital share and $\alpha_{K,t}, \alpha_{L,t}$ the factor-augmenting technologies. The equilibrium/theoretically consistent FOC's are:

[2]Klump et al. (2007, 2008) apply slightly different formulations in terms of factor shares. We have also tried this formulation and obtained almost identical results.

[3]we have also tried an iterative FGNLS estimator, which did improve the performance in the simulation but had serious convergence issues in most of the estimations.

[4]Even though the US data may be different from the Danish context, we find that changes in the calibrated values do not change the overall conclusions in any significant manner.

$$r_t^\star = \frac{\partial Y_t^\star}{\partial K_t} = \pi \alpha_{K,t}^{(\sigma-1)/\sigma} \left(\frac{Y_t^\star}{K_t}\right)^{1/\sigma} \tag{7}$$

$$w_t^\star = \frac{\partial Y_t^\star}{\partial L_t} = (1-\pi)\, \alpha_{L,t}^{(\sigma-1)/\sigma} \left(\frac{Y_t^\star}{L_t}\right)^{1/\sigma} \tag{8}$$

Simulating factor inputs and technologies. In order to measure these equilibrium outcomes (output and the factor prices), we need data on the factor inputs and the factor-augmenting technologies. We simulate capital and labor as random walks with a deterministic and a stochastic trend:

$$K_t = K_{t-1} e^{\left(\gamma_K + \varepsilon_{K,t}\right)}, \qquad L_t = L_{t-1} e^{\left(\gamma_L + \varepsilon_{L,t}\right)} \tag{9}$$

We set the initial values to $K_0 = 8$ and $L_0 = 1$ implying a K/L-ratio on 8. γ_K, γ_L are the deterministic trends and $\varepsilon_{K,t}, \varepsilon_{L,t}$ the stochastic trends. The deterministic trends are calibrated to match a constant growth rate on 1.2% in labor and 2.7% in capital, consistent with the US data. The variances of the stochastic trends are calibrated to match the variances of the growth rates of capital and labor.

The factor-augmenting technologies are likewise simulated according to a random walk with a deterministic and stochastic trend:

$$\alpha_{K,t} = \alpha_{K,t-1} e^{\left(\psi_{K,t} + \eta_{K,t}\right)}, \qquad \alpha_{L,t} = \alpha_{L,t-1} e^{\left(\psi_{L,t} + \eta_{L,t}\right)} \tag{10}$$

We try two different specifications of $\psi_{K,t}$ and $\psi_{L,t}$. The first is a constant deterministic trend, ψ_K, ψ_L. The second allows for time-variability and is calibrated based on the US data. Specifically, we apply 10-year averages from the US data implying that the level of the deterministic trend changes every tenth year in the sample. We refer to Kastrup et al. (2022) for details.

Measurement errors. In reality, factor prices are measured with error. Therefore, we add a stochastic i.i.d. error term: $r_t = r_t^\star e^{\varepsilon_t^r}$, $w_t = w_t^\star e^{\varepsilon_t^w}$. We calibrate the variance of these to match the observed GDP growth rate in the US data. In order to be consistent with national accounts, this implies that observed output is $Y_t = r_t K_t + w_t L_t$. As shown in León-Ledesma et al. (2010), this is equivalent to $Y_t = Y_t^\star \left(\eta_t e^{\varepsilon_t^r} + (1-\eta_t)\, e^{\varepsilon_t^w}\right)$, where η_t is a parameter between zero and unity. This illustrates that observed output becomes a function of the measurement errors.

3.2 Simulation results

Table 3.2.1 displays the simulation results when the deterministic trend in the factor-augmenting technologies is constant. We simulate data for four different values of the elasticity: $\sigma = 0.2, 0.5, 0.9, 1.3$. Across all values, the estimates obtained with the system estimator using both FOCs and the production function are biased toward unity. The results also show that the Box-Cox transformation of the growth rates performs better than a linear trend assumption. This is because the Box-Cox transformation is more flexible and, to a better extent, able to capture fluctuations in factor-augmenting technologies. Even so, the bias is still sizable. The same result holds true when considering the case of a time-varying deterministic trend in the factor-augmenting technologies (Table 3.2.2), where the bias increases even further. This illustrates that the Box-Cox transformation cannot capture several shifts in factor-augmenting technological change.

The system estimators are biased toward unity due to measurement errors in the factor prices. This is because, by construction, output becomes endogenous as $cov(Y_t, \varepsilon_t^r) \neq 0$. Thus, measurement errors create an endogeneity bias through output. In Kastrup et al. (2022), we show that by applying the equilibrium output instead, i.e., $Y_t^\star$, it holds that $cov(Y_t^\star, \varepsilon_t^r) = 0$, which removes the bias toward unity. However, such an exercise is impossible in observed data as we cannot separate observed output and the equilibrium outcome.

The primary invention of this paper is to apply the relative FOC instead (i.e., equation 5). This implies that output cancels out from the FOC. As evident from Table 3.2.1 and 3.2.2, this completely removes the bias. Thus, the system estimator estimating the relative FOC and the production function imply unbiased estimates.

Table 3.2.1: Simulation with constant trend in data

Estimation method	**Actual σ** 0.2	0.5	0.9	1.3
System, linear trend	0.30 (0.21;0.48)	0.64 (0.52;0.81)	0.93 (0.86;1.00)	1.12 (1.00;1.30)
Relative sys., linear trend	0.20 (0.15;0.30)	0.50 (0.40;0.64)	0.90 (0.78;1.02)	1.30 (1.06;1.74)
System, BoxCox	0.26 (0.19;0.40)	0.58 (0.47;0.75)	0.92 (0.85;0.99)	1.22 (1.05;1.41)
Relative sys., BoxCox	0.20 (0.15;0.27)	0.49 (0.40;0.64)	0.90 (0.79;1.02)	1.28 (1.07;1.68)

Notes: Point estimates taken as the median of 1.000 simulations. 5%- and 95%-quantiles are presented in parenthesis below their corresponding estimates.

Table 3.2.2: Simulation with time variant trend in data

Estimation method	**Actual σ** 0.2	0.5	0.9	1.3
System, linear trend	0.35 (0.21;0.67)	0.70 (0.51;1.00)	0.95 (0.87;1.01)	1.08 (0.98;1.28)
Relative sys., linear trend	0.20 (0.13;0.42)	0.49 (0.36;0.77)	0.91 (0.78;1.07)	1.29 (0.97;1.86)
System, BoxCox	0.31 (0.17;0.55)	0.62 (0.44;0.99)	0.94 (0.84;1.05)	1.16 (0.94;1.48)
Relative sys., BoxCox	0.20 (0.13;0.33)	0.48 (0.36;0.71)	0.92 (0.79;1.08)	1.25 (0.94;1.82)

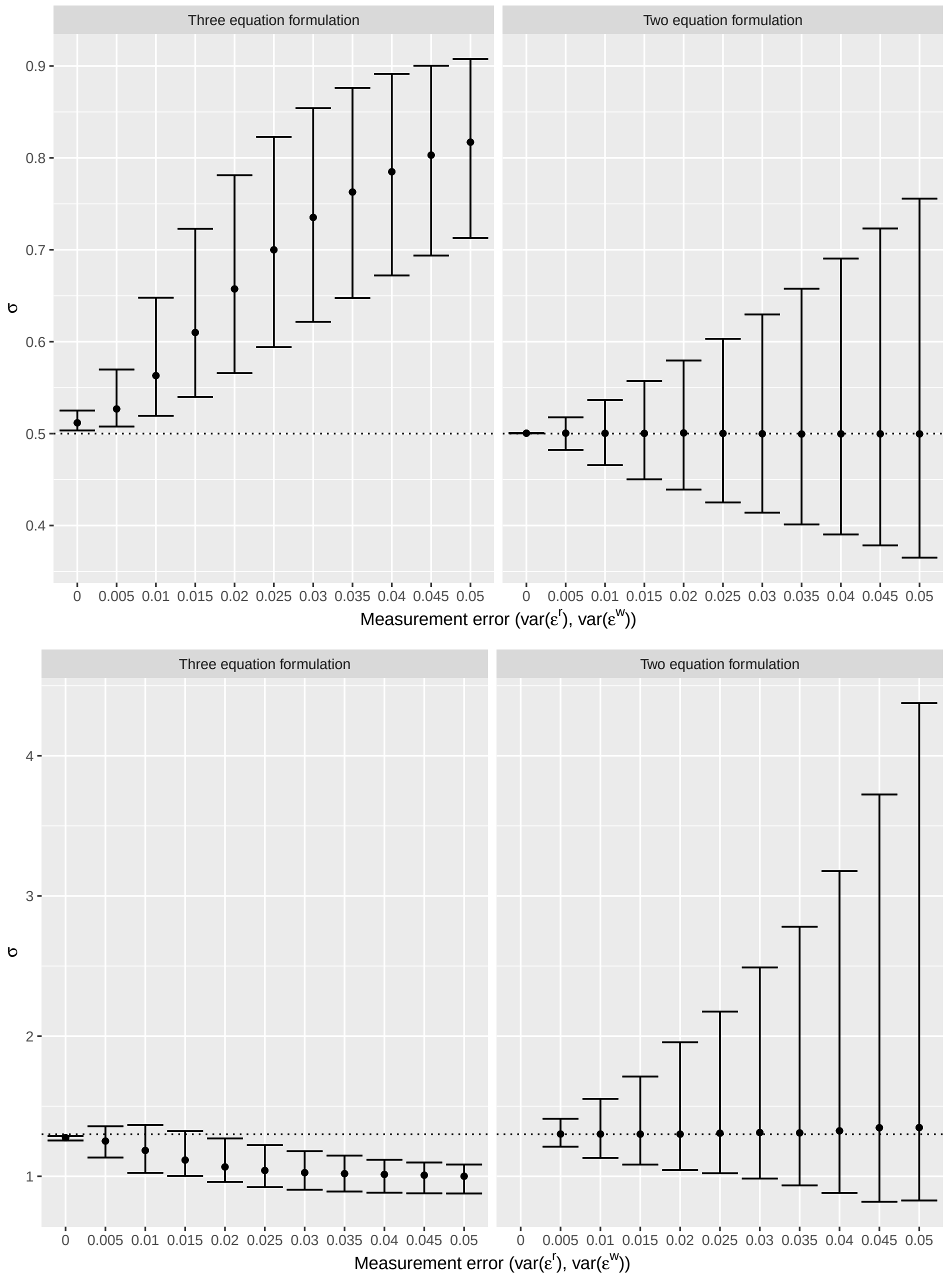
Notes: Point estimates taken as the median of 1.000 simulations. 5%- and 95%-quantiles are presented in parenthesis below their corresponding estimates.

The role of measurement errors. A bias in the estimated elasticity either originates if the true process of technical change is different from a linear trend or if factor prices are measured with error. We next illustrate the role of measurement errors in a simulation exercise. We assume that the true process of technical change is a linear trend (i.e., $\eta_t^K = \eta_t^L = 0$) and vary the degree of measurement errors, i.e., the standard deviation of $\varepsilon_t^r, \varepsilon_t^w$. We simulate data for ten different values of the measurement error of ε_t^w and impose the assumption, as before, that the measurement error in the user cost is twice the size as in the wage.

The results of the simulation exercise is displayed in Figure 3.2.1. We first simulate data for $\sigma = 0.5$ and apply both the three-equation and the two-equation system estimators. The results clearly illustrate that the three-equation system is upward biased as the degree of measurement error increases. Oppositely, the two-equation system is unbiased and increasing the measurement errors only increases the uncertainty of the estimates, but not the median estimate. We next simulate data for $\sigma = 1.3$. Opposite to before, the three-equation system is now downward biased as the degree of measurement errors increases and the two-equation system remain largely unbiased.

The simulation exercise shows that a three-equation system estimator, consisting of the two FOC and the production function is biased toward unity. In Appendix A we prove analytically that as the degree of measurement error increases, the estimates becomes increasingly biased. The direction of the bias is determined by the true elasticity of substitution.

Figure 3.2.1: Varying degree of measurement errors



Notes: Point estimates taken as the median of 1.000 simulations. 5%- and 95%-quantiles are presented in parenthesis below their corresponding estimates.

In Appendix A, we perform the opposite exercise where we set all measurement errors to zero, but let technical change deviate from a linear trend. The simulations show that the three-equation system is again biased toward unity while the two-equation system is largely unbiased as the stochastic trend of technical change increases. This illustrates that even in the presence of misspecificied technical change, the two-equation approach provide largely unbiased estimates.

4 Empirical estimations

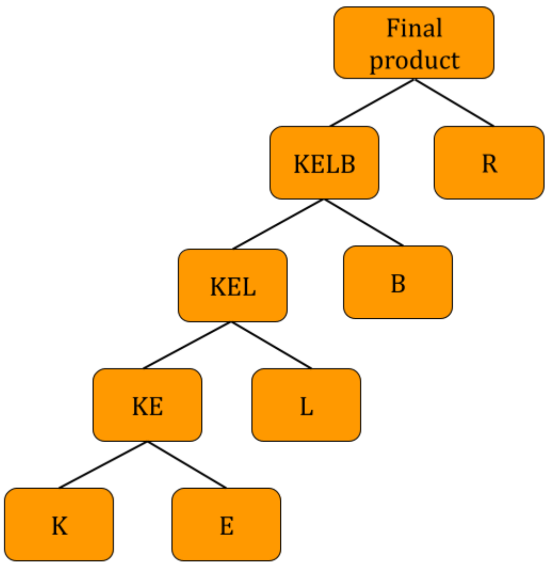
Having illustrated the performance of the new system estimator, we next update the substitution elasticities in MAKRO. In Subsection 4.1, we present the data applied in estimation and the estimated elasticities are displayed in Subsection 4.2.

4.1 Data

The data for the estimation, including a time series of user cost of capital, is gathered from MAKRO's databank supplied by Statistics Denmark (as of August 2023). Data spans from 1983 to 2019. Estimations are done sector-wise, following MAKRO sectors, i.e. Construction, Energy, Manufacturing, Agriculture, Maritime transport, Service, and Extraction (Housing and Public are omitted).

Production in MAKRO follows a nesting structure. Every nest has its own CES production function with two inputs, one of them being the aggregate-output from the previous nest-level. The input factors are equipment capital (K), energy (E), labor (L), structures capital (B) and intermediate inputs (R), nested as shown in figure 4.1.1. Each of the four production functions in this structure has its own elasticity of substitution. Note that the working paper Kronborg and Poulsen (2021) uses MAKRO's previous nesting structure where energy was part of the intermediate inputs bundle. Hence, the results in this note are not directly comparable to the results from Kronborg and Poulsen (2021).

Figure 4.1.1: MAKRO production nest structure of equipment capital (K), energy (E), labor (L), structures capital (B) and intermediate inputs (R)



The output measure in the production of KE, KEL and KELB is calculated using the known quantity of both K, E, L and B from data, as well as prices of E and L. For the price of K and B, the MAKRO user cost of equipment capital and structures capital are used, respectively.

The user cost in MAKRO is calculated from a sum of the model's Tobin's q, production tax on capital, change in installation costs, and a discounted value of change in future installation costs, minus tax shield and collateral value on capital. For a detailed derivation of MAKRO's user cost, see Bonde et al. (2023). Price and quantity series are presented in Data series.

In $t = 1983$, the price of KE, KEL and KELB are initially all set to 1, and later normalized with base year 2010, ensuring consistency with the rest of the MAKRO data. Their quantities are obtained by summing the value of the inputs in 1983. For the subsequent years up to 2019, a Paasche price index is calculated, and from that a quantity is deduced. This method assumes zero profit in the production of every nest-unit except the last, KELBR, which is the final product with a markup. The output measure of KELBR is then simply the total production in the given sector.

4.2 Estimations on Danish data

The estimation results using the System method's relative formulation are presented in table 4.2.1. Across sectors and nests, most elasticities are between 0.5-0.9. The exceptions are the capital-energy elasticity in extraction, estimated above unity, and the maritime transport sector, which is generally estimated with considerable uncertainty.

We apply two specifications of the trend process in the estimations: A linear and a Box-Cox trend. Across sectors and nests, we find minor differences between these two trend assumptions. As the estimations with the linear trend are generally estimated with higher precision, reflected by a lower standard error, we apply these estimates as the main estimates. These estimates are used in the most recent version of MAKRO.

Table 4.2.1: Estimated substitution elasticities between different aggregates of equipment capital (K), energy (E), labor (L), structures capital (B) and intermediate inputs (R)

Sector	Method	KE	(KE)L	(KEL)B	(KELB)R
Construction	Relative linear	0.73 (0.09)	0.82 (0.10)	0.92 (0.08)	0.88 (0.23)
	Relative BoxCox	0.70 (0.18)	0.82 (0.16)	0.85 (0.32)	0.89 (0.25)
Energy	Relative linear	0.88 (0.16)	0.90 (0.31)	0.57 (0.26)	0.94 (0.29)
	Relative BoxCox	0.88 (0.18)	0.88 (0.41)	0.53 (0.27)	0.94 (0.43)
Manufacturing	Relative linear	0.94 (0.29)	0.94 (0.12)	0.86 (0.14)	0.95 (0.25)
	Relative BoxCox	0.93 (0.38)	0.92 (0.23)	0.86 (0.38)	0.95 (0.29)
Agriculture	Relative linear	0.95 (0.64)	0.87 (0.10)	0.93 (0.37)	0.90 (0.60)
	Relative BoxCox	0.95 (0.76)	0.87 (0.19)	0.93 (0.44)	0.89 (0.30)
Maritime transport	Relative linear	113.82 (1672.76)	0.97 (0.19)	0.97 (0.14)	12.32 (26.83)
	Relative BoxCox	43.96 (459.43)	0.96 (0.21)	0.97 (0.18)	4.01 (3.27)
Services	Relative linear	0.94 (0.15)	0.81 (0.09)	0.55 (0.16)	0.94 (0.26)
	Relative BoxCox	0.94 (0.12)	0.81 (0.16)	0.43 (0.17)	0.94 (0.45)
Extraction	Relative linear	1.72 (0.23)	0.84 (0.06)	0.97 (0.13)	0.86 (0.20)
	Relative BoxCox	1.84 (0.62)	0.65 (0.19)	0.97 (0.24)	0.87 (0.22)

Notes: Estimations are done with the relative formulated system method. Standard errors are presented in parenthesis below their corresponding estimates.

5 Concluding remarks

In this paper, we have presented a new system estimator approach to estimate the elasticity of substitution between production factors, addressing issues related to measurement errors in the user cost of capital. Through simulations and empirical application to MAKRO data, we have demonstrated the effectiveness of our method in providing unbiased estimates of the elasticities, even in the presence of misspecified technical change. Our results indicate that the estimated elasticities across various sectors do not suffer from the downward bias observed in previous studies. This paper contributes to the ongoing research on macroeeconomic modeling by offering a robust framework for elasticity estimation.

References

Bonde, M., Ejarque, J., Hoegh, G., Partsch, E., Stephensen, P., and Vasi, T. (2023). Makro model documentation. pages 47–50. 4.1

Diamond, P., McFadden, D., and Rodriquez, M. (1978). *Measurement of the Elasticity of Factor Substitution and Bias of Technical Change*, volume 2, chapter 5. Production Economics: A Dual Approach to Theory and Applications. 1

Hulten, C. R. (2010). *Growth Accounting*, volume 2, chapter 23, pages 987–1031. Handbook of the Economics of Innovation. 1

Karabarbounis, L. and Neiman, B. (2014). The global decline of the labor share. *Quarterly Journal of Economics*, 129(1):61–104. 1

Kastrup, C. B., Kronborg, A. F., and Stephensen, P. P. (2022). Estimation of the elasticity of substitution. Technical report, DREAM working paper. 1, 3.1, 3.1, 3.2

Klump, R., McAdam, P., and Willman, A. (2007). Factor substitution and factor-augmenting technical progress in the united states: A normalized supply-side system approach. *Review of Economics and Statistics*, 89(1):183–192. 2, 2

Klump, R., McAdam, P., and Willman, A. (2008). Unwrapping some euro area growth puzzles: Factor substitution, productivity and unemployment. *Journal of Macroeconomics*, 30(2):645–666. 2

Kronborg, A. F. and Poulsen, K. A. (2021). Estimater for elasticiteterne i makros produktionsfunktion. Technical report, DREAM working paper. 1, 4.1

Kronborg, A. F., Poulsen, K. A., and Kastrup, C. B. (2021). Estimating ces production functions in makro. Technical report, DREAM working paper. 1

León-Ledesma, M. A., McAdam, P., and Willman, A. (2010). Identifying the elasticity of substitution with biased technical change. *American Economic Review*, 100(4):1330–1357. 1, 2, 2, 3.1

León-Ledesma, M. A., McAdam, P., and Willman, A. (2015). Production technology estimates and balanced growth. *Oxford Bulletin of Economics and Statistics*, 77(1):40–65. 2

Oberfield, E. and Raval, D. (2021). Micro data and macro technology. *Econometrica*, 89(2):703–732. 1

A Measurement errors

The bias of a OLS estimator is given by

$$\hat{\beta} - \beta = \frac{cov\left(X_t, \varepsilon_t\right)}{var\left(X_t\right)}$$

Assume that the only source of measurement error is through capital, $\varepsilon_t^r \neq 0, \varepsilon_t^w = 0$. In that case, the estimation equation of capital is:

$$\log\left(r_t\right) = \log\left(\bar{\pi}\frac{\bar{Y}}{\bar{K}}\right) + \frac{1}{\sigma}\log\left(\frac{Y_t/\bar{Y}}{K_t/\bar{K}}\right) + \frac{\sigma-1}{\sigma}\left(\log\left(\xi\right) + \psi_K\left(t-\bar{t}\right)\right) + \hat{\varepsilon}_t^r$$

$$\hat{\varepsilon}_t^r = \varepsilon_t^r - \frac{1}{\sigma}\log\left(\eta_t e^{\varepsilon_t^r} + \left(1-\eta_t\right)\right) = \frac{\sigma-1}{\sigma}\varepsilon_t^r$$

Remembering that $Y_t = Y_t^\star\left(\eta_t e^{\varepsilon_t^r} + \left(1-\eta_t\right)e^{\varepsilon_t^w}\right)$, the bias of the estimated elasticity becomes:

$$\frac{1}{\hat{\sigma}} - \frac{1}{\sigma} = \frac{cov\left(\log Y_t, \hat{\varepsilon}_t^r\right)}{var\left(\log Y_t\right)} = \frac{\sigma-1}{\sigma}\frac{var\left(\varepsilon_t^r\right)}{var\left(Y_t^\star\right) + var\left(\varepsilon_t^r\right)}$$

When $var(\varepsilon_t^r) = 0$, it implies that $\hat{\sigma} = \sigma$. When $var\left(\varepsilon_t^r\right) > 0$, the estimate is biased. The direction of the bias is determined by the value of the true elasticity. When $\sigma < 1$, it holds true that $\frac{1}{\hat{\sigma}} - \frac{1}{\sigma} < 0 \rightarrow \hat{\sigma} > \sigma$. Oppositely, when $\sigma > 1$, $\frac{1}{\hat{\sigma}} - \frac{1}{\sigma} > 0 \rightarrow \hat{\sigma} < \sigma$. Thus, the more the user cost is measured with error, the larger is the bias.

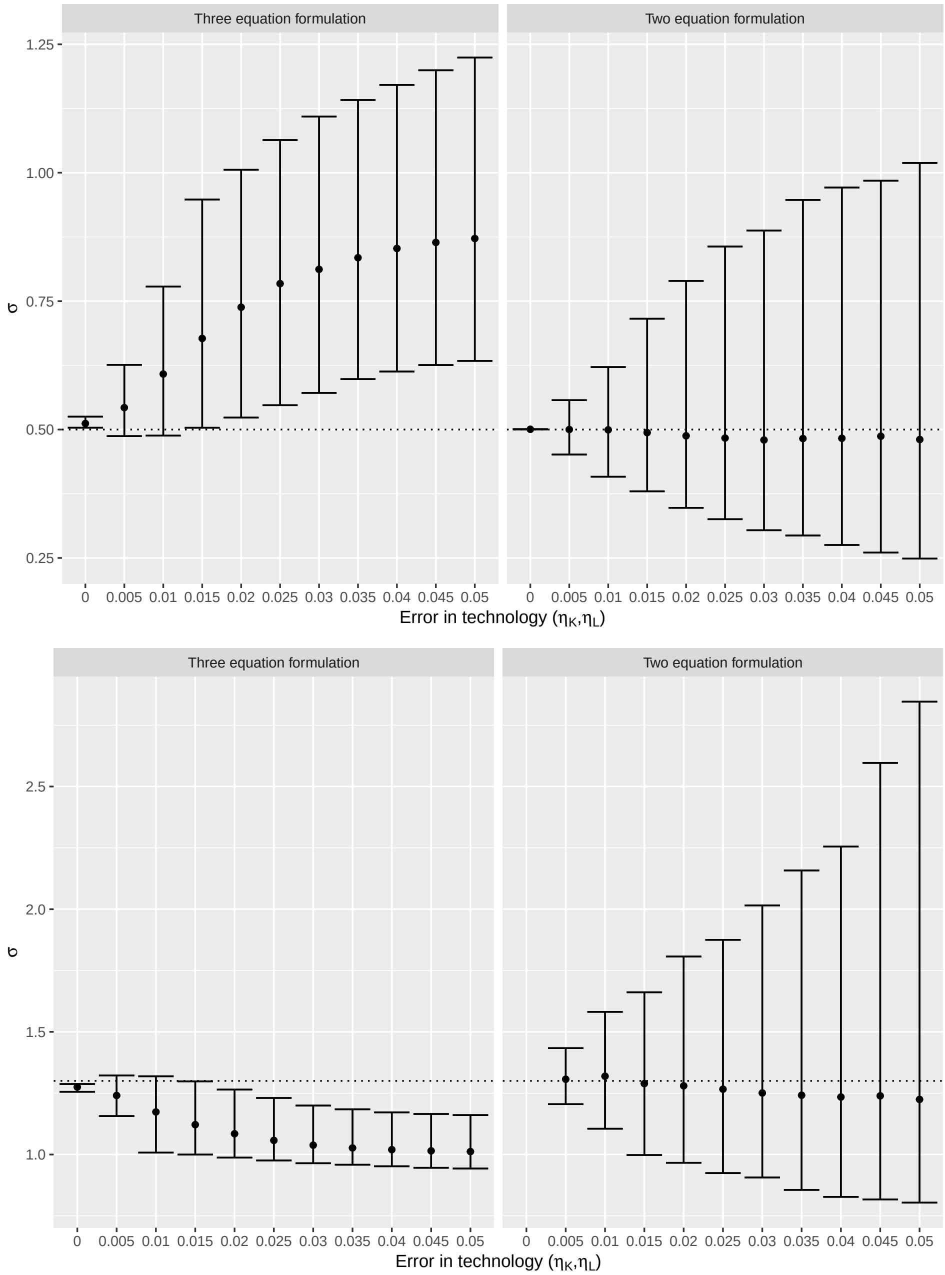
It is easy to illustrate that this bias vanishes when applying the relative FOC:

$$\log\left(r_t/w_t\right) = \log\left(\frac{\bar{\pi}}{1-\bar{\pi}}\frac{\bar{L}}{\bar{K}}\right) + \frac{1}{\sigma}\log\left(\frac{L_t/\bar{L}}{K_t/\bar{K}}\right) + \frac{\sigma-1}{\sigma}\left(\psi_K - \psi_L\right)\left(t-\bar{t}\right) + \hat{\varepsilon}_t$$

$$\hat{\varepsilon}_t = \varepsilon_t^r - \varepsilon_t^w$$

As L_t and K_t are uncorrelated with the measurement error, no endogeneity bias exists.

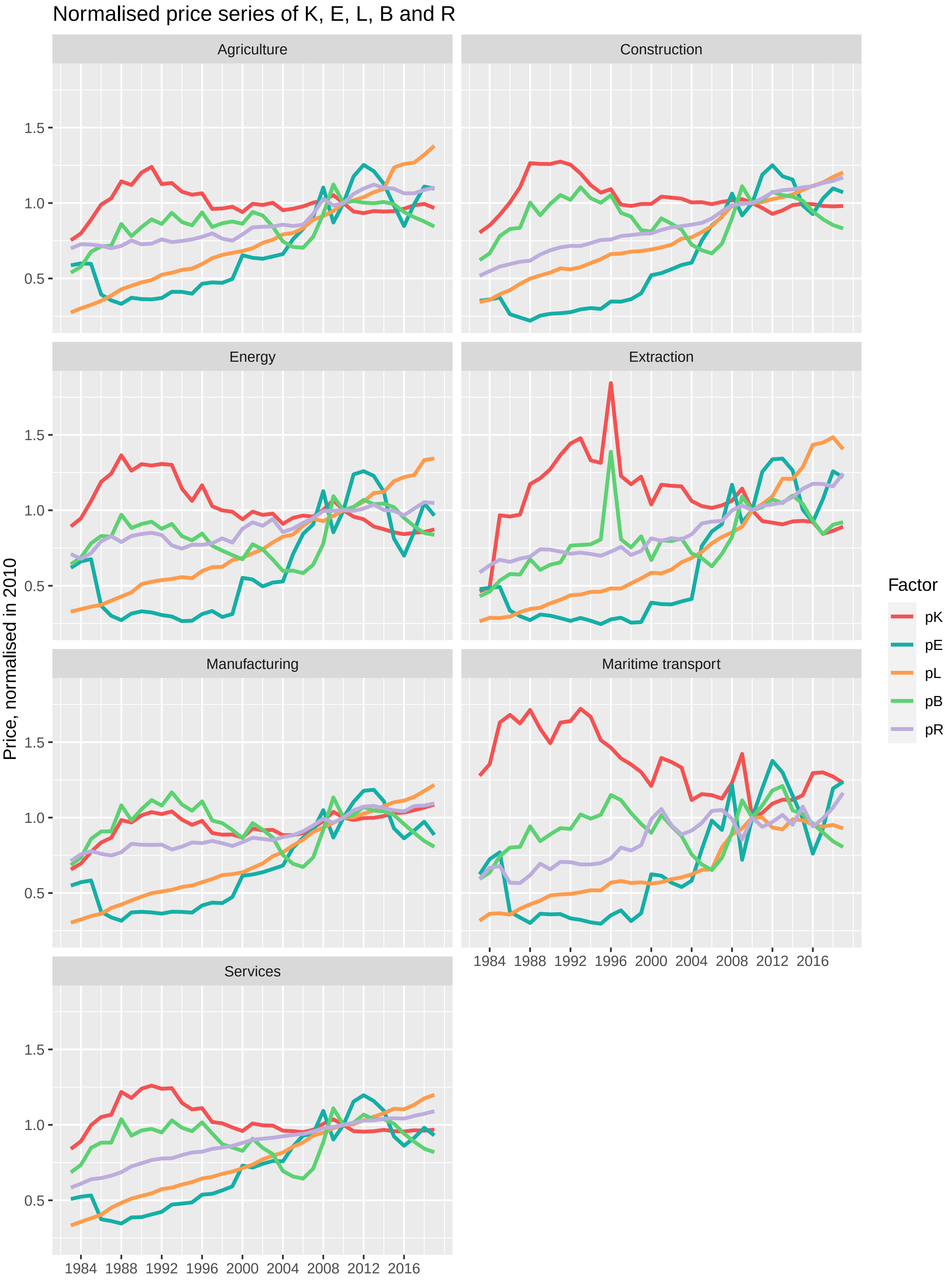
Table A.0.1: Varying degree of technology errors



Notes: Point estimates taken as the median of 1.000 simulations. 5%- and 95%-quantiles are presented as the bars in the plot.

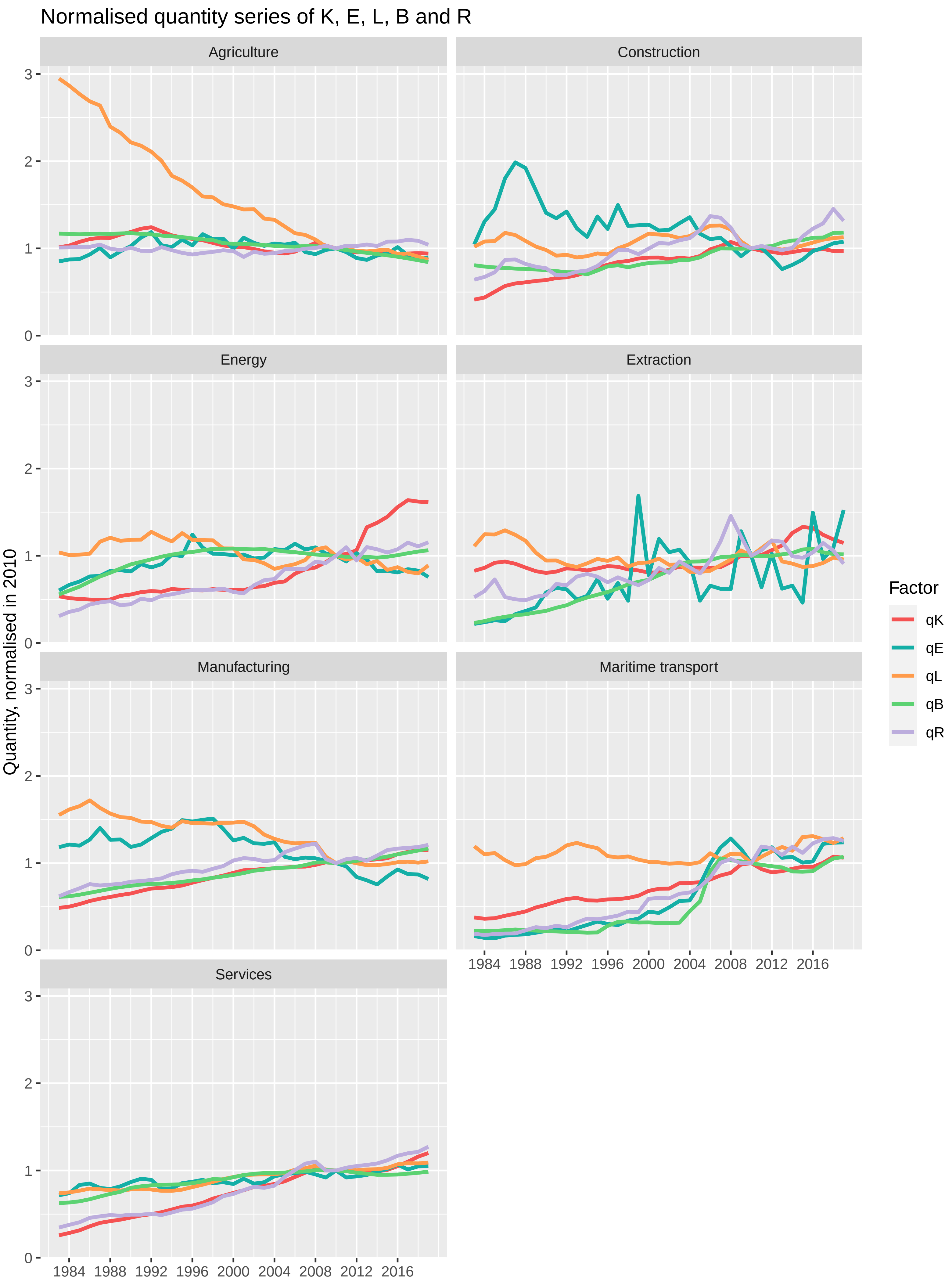
B Data series

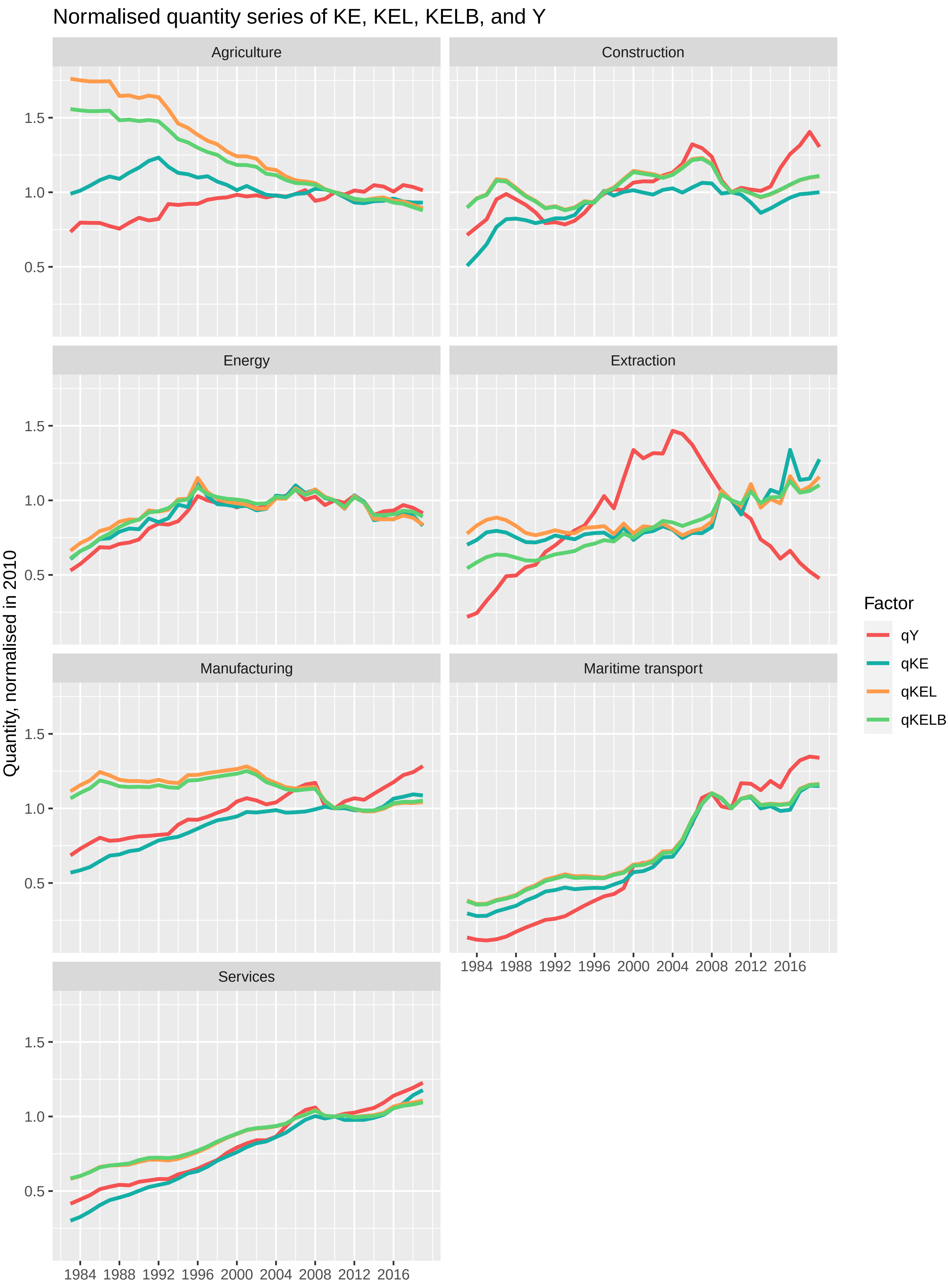
Normalised price and quantity series used in the empirical estimation are shown on the following pages.



Normalised price series of K, E, L, B and R
Agriculture
Construction
Energy
Extraction
Manufacturing
Maritime transport
Services
Price, normalised in 2010
1.5
1.0
0.5
1984
1988
1992
1996
2000
2004
2008
2012
2016
Factor
pK
pE
pL
pB
pR


Normalised price series of KE, KEL, KELB, and Y
Agriculture
Construction
Energy
Extraction
Manufacturing
Maritime transport
Services
Price, normalised in 2010
0.4
0.8
1.2
1.6
1984
1988
1992
1996
2000
2004
2008
2012
2016
Factor
pY
pKE
pKEL
pKELB


Normalised quantity series of K, E, L, B and R
Agriculture
Construction
Energy
Extraction
Manufacturing
Maritime transport
Services
Quantity, normalised in 2010
0
1
2
3
1984
1988
1992
1996
2000
2004
2008
2012
2016
Factor
qK
qE
qL
qB
qR



Normalised quantity series of KE, KEL, KELB, and Y
Agriculture
Construction
Energy
Extraction
Manufacturing
Maritime transport
Services
Quantity, normalised in 2010
0.5
1.0
1.5
1984
1988
1992
1996
2000
2004
2008
2012
2016
Factor
qY
qKE
qKEL
qKELB